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NUMBER THEORY

Lalan, V. Sur certains nombres figurés lacunaires. Mathesis **58**, 77–79 (1949).

Suppose the integers $1, 2, 3, \cdots$ to be written in a row. Choose any integer m as "lacunary (gap) index" and consider the multiples of m in the row to be suppressed. Under each of the remaining integers place the sum of the integer and those preceding it. In the second row consider every $(m-1)$th integer to be suppressed and proceed as above. In the third row suppress every $(m-2)$th integer and continue as before. The configuration formed by the remaining integers consists of a series of right triangles. The integers of the final row are the mth powers of the integers $1, 2, 3, \cdots$. Call the triangles $T_0, T_1, T_2, \cdots$, and let $L^{\lambda}_{k,n}$ be the integer in the kth row and nth column of the triangle T_λ. Then

$$L^{\lambda}_{k,n} = F_{k,n} + C_p^1 F_{k-1,n}\lambda + \cdots + C_p^r F_{k-r,n}\lambda^r + \cdots + C_p^k F_{0,n}\lambda^k,$$

where the C_p^r are the binomial coefficients and $F_{k,n}$ the nth ordinary figurative number of order k. In particular, if the integer in the kth row on the hypotenuse of triangle T_n is divided by the corresponding integer of T_0, the result is n^k. Numerical examples are given for $m=2, 3, 4$.

W. H. Simons (Vancouver, B. C.).

Chater, N., and Chater, W. J. On the determinants of pan-magic squares of even order. Math. Gaz. **33**, 94–98 (1949).

Announcement is made of a forthcoming proof that in a pan-magic (pan-diagonal, diabolic, Nasik) magic square of even order $2N$ with arbitrary elements, complementary elements (whose sum is one-Nth the magic sum) occupy positions separated diagonally by $N-1$ elements. This property is used to prove that every such square has determinant zero. *R. J. Walker* (Ithaca, N. Y.).

Thébault, Victor. Concerning two classes of remarkable perfect square pairs. Amer. Math. Monthly **56**, 443–448 (1949).

Huybregts W. P., J. On decomposition into factors. Simon Stevin **26**, 200–202 (1949). (Dutch)

An algorithm is described for the determination of divisors d of the integer n, of the form $d \equiv a \pmod{v}$; n, v and a are given. In some cases the method is slightly superior to the trivial method. *N. G. de Bruijn* (Delft).

Davidek, Richard. Über die Zerlegung grosser Zahlen in Faktoren. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. **85**, 248–253 (1948).

An informal discussion of the well-known method of factoring numbers by representing them as differences of two squares with the aid of small excluding moduli and movable strips of paper. The advantages to be gained by breaking the problem into two or more parts are discussed. A stencil method is alluded to but not described. *D. H. Lehmer.*

Gambelli, Lucio. Sui caratteri di divisibilità. Con una tabella dei coefficienti di divisibilità di tutti i numeri da 2 a 101. Period. Mat. (4) **27**, 109–116 (1949).

The author points out the fact that a given number N is congruent modulo m to a linear combination of the decimal digits of N in which the coefficients $c_n \equiv 10^n \pmod{m}$ depend only on m. In particular, criteria for the divisibility of N by m are thus obtained. Tables of c_n are given for all values of n and for $m \leqq 101$. Absolutely least residues modulo m are tabulated. *D. H. Lehmer* (Berkeley, Calif.).

Katz, Alexander. Some more new factors of Fibonacci-numbers. Riveon Lematematika **3**, 54 (1949).

Summary of the author's paper in Hebrew, same vol., 14 (1949); these Rev. **10**, 681.

Jarden, Dov. Recurring sequences. Math. Student **16** (1948), 28–30 (1949).

Consider the recursion relation $U_n = \sum_{i=1}^{s} a_i U_{n-s+i-1}$ for fixed integers $s \geqq 2$ and a_i, n being any integer. Let $U(n)$ denote the sequence $U_n, U_{n+1}, \cdots, U_{n+s-2}$. The principal results are for the case where each member of $U(0)$ is 0, and $U_{s-1}=1$. It is proved that any common divisor of the sequence $U(n)$ is a divisor of each member of $U(kn)$ for any positive integer k. Given any nonzero q there exists a least n, called the rank of q, such that q divides each member of $U(n)$. If q divides each of $U(N)$, then n divides N. It is the reviewer's opinion that the proof of this last result is incorrect, since it is assumed at one point that U_n is integral when n is negative. The following appears to be a counter-example to the result: let $a_i = q > 1$ for $i = 1, 2, \cdots, s$; then the rank of q is s; also q divides each member of $U(n)$ for all $n \geqq s$. *I. Niven* (Eugene, Ore.).

Jarden, Dov. Disjunctive sequences. Riveon Lematematika **3**, 54 (1949).

Summary of the author's paper in Hebrew, same vol., 15–18 (1949); these Rev. **10**, 681.

Zakay, Shlomo. Simple proof of Wolstenholme's theorem. Riveon Lematematika **3**, 58–59 (1949). (Hebrew. English summary)

Bagchi, Haridas. Note on the two congruences

$$ax^2+by^2+c \equiv 0, \quad ax^2+by^2+cz^2+dw^2 \equiv 0 \pmod{p},$$

where p is an odd prime and

$$a \not\equiv 0, \quad b \not\equiv 0, \quad c \not\equiv 0, \quad d \not\equiv 0 \pmod{p}.$$

Rend. Sem. Mat. Univ. Padova **18**, 311–315 (1949).

It is pointed out that the existence of solutions of the first congruence mentioned in the title can be demonstrated by the method usually applied to the special case where $a=b=c=1$. It follows that the second congruence mentioned is also solvable nontrivially. *I. Niven.*

Rosenthall, E. Multiplicative Diophantine equations in quaternions. Amer. J. Math. 71, 791–799 (1949).

All solutions of $XY=ZW$ in Lipschitz integral quaternions are obtained in terms of 12 quaternion parameters and 2 rational integral parameters. The solutions as given are not explicit because two pairs of the quaternion parameters are required to have equal norms. However, another theorem gives all solutions of $X\bar{X}=Y\bar{Y}$ in terms of 3 quaternion parameters. The proof of the principal result employs a theorem of Pall [Trans. Amer. Math. Soc. 47, 487–500 (1940); these Rev. 2, 36] giving conditions for quaternions to have common divisors. *I. Niven* (Eugene, Ore.).

Whiteman, Albert Leon. Theorems analogous to Jacobsthal's theorem. Duke Math. J. 16, 619–626 (1949).

The sums $\phi_q(s)=\sum_{m=1}^{p-1}(m/p)((m^q+s)/p)$, where (m/p) denotes the quadratic character of m mod p, were introduced by E. Jacobsthal [J. Reine Angew. Math. 132, 238–245 (1907)], and the solutions a, b of $a^2+b^2=p$ were expressed in terms of these sums for $p=4f+1$. This was extended by von Schrutka [J. Reine Angew. Math. 140, 252–265 (1911)], and by Chowla [Proc. Lahore Philos. Soc. 7, 2 pp. (1945); these Rev. 7, 243] to the equation $a^2+3b^2=p$, $p=3f+1$. The present paper obtains these results, and in addition expresses the solutions of $a^2+50b^2+50c^2+125d^2=16p$, $p=5f+1$ and $a^2+2b^2=p$, $p=8f+1$ in terms of the Jacobsthal sums. For example, in the latter case $a=\frac{1}{4}\phi_4(r)$, where r is any biquadratic residue of p. The proofs are based on a result relating the Jacobsthal sums and cyclotomic numbers. *I. Niven* (Eugene, Ore.).

Whiteman, Albert Leon. Theorems on quadratic residues. Math. Mag. 23, 71–74 (1949).

Dirichlet proved that if p is a prime of the form $4j+1$ then among the integers $1, 2, 3, \cdots, (p-1)/2$ there are more quadratic residues than nonresidues. Hence

$$(1)\qquad \sum_{n=1}^{(p-1)/2}\left(\frac{n}{p}\right)\gtrless 0$$

according as $p\equiv 3$ or $1 \pmod 4$, $\left(\frac{n}{p}\right)$ being the Legendre symbol. In the present paper the author establishes the following identities:

$$(2)\qquad \tfrac{1}{2}p^{-1}\sum_{m=1}^{p-1}\cot \pi m^2/p=\sum_{n=1}^{(p-1)/2}[n^2/p]-(p-1)(p-5)/24,$$

$$(3)\qquad \sum_{m=1}^{p-1}\cot \pi m^2/p=-2p^{-1}\sum_{n=1}^{p-1}n\left(\frac{n}{p}\right),$$

where $[x]$ denotes the greatest integer not exceeding x. The proofs of (2) and (3) depend upon the well-known Fourier expansion for $((x))=x-[x]-\frac{1}{2}$ and classical results about Gauss sums. Also it is shown that if p is an odd prime then

$$(4)\qquad \sum_{n=1}^{(p-1)/2}\left(\frac{n}{p}\right)=-p^{-1}\left\{2-\left(\frac{2}{p}\right)\right\}\sum_{n=1}^{p-1}n\left(\frac{n}{p}\right).$$

By means of the results (1), (2), (3), (4) the author shows that

(a) $\sum_{m=1}^{p-1}\cot \pi m^2/p\gtrless 0$ according as $p\equiv 3$ or $1 \pmod 4$,

and

(b) $$\sum_{n=1}^{(p-1)/2}\left[\frac{n^3}{p}\right]\gtrless (p-1)(p-5)/24,$$

according as $p\equiv 3$ or $1 \pmod 4$. *W. H. Simons.*

Ankeny, N. C., and Chowla, S. The class number of the cyclotomic field. Proc. Nat. Acad. Sci. U. S. A. 35, 529–532 (1949).

The class number of the field $P(1^{1/l})$ ($l>2$ prime) can be expressed as a product of two integer factors, the second factor being the class-number of the greatest real subfield of $P(1^{1/l})$. Kummer [J. Math. Pures Appl. (1) 16, 377–498 (1851), in particular, p. 473] stated, "La loi asymptotique des valeurs de ce premier facteur du nombre des classes H est exprimée par la formule $P'=l^{\frac{1}{2}(l+1)}2^{\frac{1}{2}(3-l)}\pi^{\frac{1}{2}(1-l)}$, dont je me réserve la demonstration. . . ." The authors prove $\log P'=\frac{1}{4}(l+3)\log l+\frac{1}{2}(3-l)\log 2+\frac{1}{2}(1-l)\log \pi+o(\log d)$. The proof is based on two theorems on the distribution of primes in arithmetical progressions by Titchmarsh [Rend. Circ. Mat. Palermo 54, 414–429 (1930)] and Walfisz [Math. Z. 40, 592–607 (1935)]. *H. Heilbronn* (Bristol).

Šafarevič, I. R. A general reciprocity law. Doklady Akad. Nauk SSSR (N.S.) 64, 25–28 (1949). (Russian)

It is well known that the general reciprocity law in algebraic number fields reduces to the problem of evaluating the p^mth power norm residue symbols $\left(\frac{\alpha,\beta}{\mathfrak{p}}\right)$, where $\mathfrak{p}\mid p$. The author states that α and β can be represented in the form $\pi^a w\prod E(\alpha_i,\pi^i)\zeta^{p^mB}$, where π is a local uniformizing parameter, w a root of unity of order prime to p, $E(\alpha,x)$ a certain power series in x, ζ^{p^mB} a p^mth primary number, and i runs through a set of integers relatively prime to p. Using this decomposition a p^mth primary number ζ^{p^mA} is defined in such a way that $\left(\frac{\alpha,\beta}{\mathfrak{p}}\right)=\zeta^{S(a)}$, where $a=A^p-A$ and $S(a)$ is the trace of a [cf. H. Hasse, J. Reine Angew. Math. 176, 174–183 (1936)]. No proofs are given. *W. H. Mills* (New Haven, Conn.).

Šafarevič, I. R. A general reciprocity law. Mat. Sbornik N.S. 26(68), 113–146 (1950). (Russian)

Let k be an arbitrary discrete valuated field of characteristic zero containing the primitive p^nth roots of unity. Let the residue class field of k be perfect. Then if λ and μ are arbitrary nonzero elements of k, a p^nth-primary number (λ,μ) is defined using the methods of the paper reviewed above; it is a bilinear anti-symmetric function of λ and μ, closely related to the norm residue symbol. The results stated in the previous paper are proved for the case $p\neq 2$, $n=1$. An apparent disagreement between the results of the two papers is due to a misprint in the first paper. *W. H. Mills* (New Haven, Conn.).

Min, Szu-Hoa. Euclidean algorithm in real quadratic fields. Sci. Rep. Nat. Tsing Hua Univ. Ser. A. 5, 190–225 (1948).

While the result of this paper pertaining to the nonexistence of the Euclidean algorithm in certain quadratic fields was interesting at the time the paper was written, this result has now been superseded and the whole question settled. See the review of a paper by K. Inkeri [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. 41 (1947); these Rev. 10, 15]. *L. Schoenfeld* (Urbana, Ill.).

Maass, Hans. Automorphe Funktionen und indefinite quadratische Formen. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1949, no. 1, 42 pp. (1949).

Set $\Delta=y^2(\partial^2/\partial x^2+\partial^2/\partial y^2)$, let k be a nonnegative integral rational number and r a real parameter. If, in the half plane

$y>0$, the $(2k+2)$-fold differentiable function $g(\tau)$ satisfies the system of differential equations:

$$\prod_{\nu=0}^{k}(\Delta-(r+\nu)(r+\nu-1))\tau^{\mu}g(\tau)=0, \qquad \mu=0,1,\cdots,k,$$

as well as the relationship $g(S\tau)=v(S)(c\tau+d)^k g(\tau)$ for S an element of the group G of real 2 by 2 unimodular substitutions, $v(s)$ having the properties: $v(S_1)v(S_2)=v(S_1S_2)$ and $v(-E)=(-1)^k$, where E is the identity matrix, then $g(\tau)$ is called a "wave form" [Wellenform] of dimension $-k$ and parameter r. The principal theorem consists of a relationship between automorphic wave forms of dimension -1 and Dirichlet series in the form of a general theorem on the solutions of functional equations of the following type:

$$\xi(s)=\left\{\frac{\lambda}{\pi}\right\}^s\Gamma\left\{\frac{s+r+\frac{1}{2}}{2}\right\}\Gamma\left\{\frac{s-r-\frac{1}{2}}{2}\right\}\varphi(s), \qquad \varphi(s)=\sum_{n\neq 0}a_n/|n|^s,$$

$\eta(s)$ obtained from the above by replacing $r+\frac{1}{2}$ by $r-\frac{1}{2}$ and a_n by $a_n \operatorname{sgn} n$, $\xi(s)=\pm\eta(2-s)$. An application of this result is made to Siegel's

$$\zeta(S,s)=\sum_{t=1}^{\infty}M(S,t)t^{-s},$$

where M is the mass of the indefinite quadratic form S.

B. W. Jones (Boulder, Colo.).

Igusa, Jun-ichi. On the theory of algebraic correspondences and its application to the Riemann hypothesis in function-fields. J. Math. Soc. Japan **1**, 147–197 (1949).

The author presents in this paper his elaboration of the specified outline for the proof of the Riemann hypothesis of function fields as presented by A. Weil [C. R. Acad. Sci. Paris **210**, 592–594 (1940); Proc. Nat. Acad. Sci. U. S. A. **27**, 345–347 (1941); Foundations of Algebraic Geometry, Amer. Math. Soc. Colloquium Publ., v. 29, New York, 1946; these Rev. **2**, 123, 345; **9**, 303]. The author's details were arrived at independently and differ somewhat from Weil's final presentation [Sur les courbes algébriques et les variétés qui s'en déduisent, Actual. Sci. Ind., no. 1041, Hermann, Paris, 1948 = Publ. Inst. Math. Univ. Strasbourg **7** (1945); these Rev. **10**, 262]; e.g., only the Riemann-Roch theorem for nonspecial linear series is used. Furthermore Schubert's formula is developed in full detail and the Picard number for products of curves is discussed. *O. F. G. Schilling.*

Korobov, N. M. Some problems on the distribution of fractional parts. Uspehi Matem. Nauk (N.S.) **4**, no. 1(29), 189–190 (1949). (Russian)

The following theorems are stated. (I) Let $f(x)=\sum_{\nu=0}^{\infty}a_\nu x^\nu$, where $|a_\nu|=e^{-\omega(\nu)}$. If, for every sufficiently large ν, $\omega(\nu)>\nu^\lambda$ and $(1+\beta_1/\nu)\omega(\nu)\leqq\omega(\nu+1)\leqq\beta_2\nu\omega(\nu)$, where $\lambda>3$, $\beta_1>1$, $0<\beta_2<1$, then $f(x)$ is uniformly distributed. If a function $g(x)$ possesses the property that for every set of integers $m_1,\cdots,m_s$ not all zero the function

$$F_s(x)=m_1g(x+1)+\cdots+m_sg(x+s)$$

is uniformly distributed it is said to be completely uniformly distributed. The function $f(x)$ is such a function. (II) For every integer $q\geqq 2$ put $\alpha=\sum_{k=1}^{\infty}[\theta_k q^k]q^{-k}$, where $\theta_k=\{\varphi(k)\}$ is the fractional part of a completely uniformly distributed function $\varphi(k)$. Then αq^s is uniformly distributed. (III) Let $q_1, q_2, \cdots$ be a sequence of integers with $2\leqq q_1\leqq q_2\leqq\cdots$, $q_s\to\infty$ as $x\to\infty$, and put $f(x)=\alpha q_1q_2\cdots q_s$. Then a necessary and sufficient condition for $f(x)$ to be uniformly distributed is that α may be represented in the form $\alpha=\sum_{k=1}^{\infty}[\theta_kq_k]/(q_1q_2\cdots q_k)$, where $\theta_k=\{\varphi(k)\}$ is the fractional part of a uniformly distributed function $\varphi(k)$.

R. A. Rankin (Cambridge, England).

Gel'fond, A. O. The approximation of algebraic numbers by algebraic numbers and the theory of transcendental numbers. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 19–49 (1949). (Russian)

The author reports on the history and recent progress of the theory of transcendental numbers, laying particular stress on the work by Russian mathematicians beginning with Euler. After an introductory paragraph, the following subjects are discussed. [2] Liouville's theorem on the approximation of algebraic numbers, and its improvement by Thue and Siegel. [3] The author's recent further improvement of the Thue-Siegel theorem [similar to, but more general than, Dyson's work in Acta Math. **79**, 225–240 (1947); these Rev. **9**, 412; see also T. Schneider, Arch. Math. **1**, 288–295 (1949); these Rev. **10**, 592] in a paper unavailable outside the USSR, and its connection with the problem of effectiveness, i.e., of finding bounds for the solutions and not only for their number. The Thue-Siegel method does not have this property, but more recent work by N. Fel'dman and the author [also unavailable] is said to be promising in this direction. [4] Transcendency of e and π. Sketch of a proof of Lindemann's theorem without use of Hermite's explicit formulae. Siegel's work on Bessel functions. Measures of transcendency of e and π, in particular the new results by Fel'dman [see the second following review]. [5] Transcendency of α^β and $(\log\alpha)/(\log\beta)$ for algebraic α, β. The author's earlier result for e^π, and the final complete solution by him and Schneider. Sketch of his proof. Schneider's work on elliptic functions. The author's work on measures of transcendency of α^β and $(\log\alpha)/(\log\beta)$; its connection with the problem of effectiveness [C. R. (Doklady) Acad. Sci. URSS (N.S.) **7** (**1935 II**), 177–182; Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] **1939**, 509–518; Rec. Math. [Mat. Sbornik] N.S. **7**(**49**), 7–25 (1940); these Rev. **1**, 295, 292]. Sketch of a new method of the author by which he proves that if $a\neq 0$, $\neq 1$ is algebraic, and α is a cubic irrational, then a^α and a^{α^2} are algebraically independent over the rational field. Outlook on further applications of this method, e.g., to improve still more the measure of transcendency of $(\log\alpha)/(\log\beta)$. [6] Expressions defined by series or products, and especially the work by Morduchay-Boltovskoy. The paper ends with a bibliography. *K. Mahler* (Manchester).

Gel'fond, A. O. On the algebraic independence of transcendental numbers of certain classes. Uspehi Matem. Nauk (N.S.) **4**, no. 5(33), 14–48 (1949). (Russian)

By means of a method similar to one used by him already previously [Doklady Akad. Nauk SSSR (N.S.) **64**, 277–280 (1949); these Rev. **10**, 682], the author proves the following three theorems. (1) Let $\eta_0, \eta_1, \eta_2, \alpha_1, \alpha_2$ be real or complex numbers such that η_0, η_1, η_2, as well as 1, α_1, α_2, are linearly independent over the rational field. Let there be two positive constants τ, x' such that $|x_0\eta_0+x_1\eta_1+x_2\eta_2|>\exp(-\tau x\log x)$ for all rational integers x_0, x_1, x_2 satisfying

$$x=\max(|x_0|, |x_1|, |x_2|)>x'.$$

Then at least two of the 11 numbers

(a) $\qquad \alpha_1, \alpha_2, \exp(\eta_i\alpha_k), \qquad i=0,1,2; k=0,1,2; \alpha_0=1,$

are algebraically independent over the rational field.

(2) Let $\eta_0, \eta_1, \alpha_1, \alpha_2$ be real or complex numbers such that $\eta_0 \neq 0$, η_1/η_0 is irrational, and $1, \alpha_1, \alpha_2$ are linearly independent over the rational field. Let there be two positive constants τ, x' such that $|x_0+x_1\eta_1/\eta_0| > \exp(-\tau x^2 \log x)$ for all rational integers x_0, x_1 satisfying $x=\max(|x_0|, |x_1|)>x'$. Then at least two of the 10 numbers

$$\eta_0, \eta_1, \alpha_1, \alpha_2, \exp(\eta_i\alpha_k), \qquad i=0,1;\ k=0,1,2;\ \alpha_0=1,$$

are algebraically independent over the rational field.

(3) Let a, b, α, β be algebraic real or complex numbers such that a is different from 0 and 1, and that b and $(\log\alpha)/(\log\beta)$ are irrational. Then there exists to every $\epsilon>0$ a number H_0 with the following properties. If $P(x)\neq 0$ is a polynomial of degree at most s and with rational integral coefficients of absolute value at most H, then

$$|P(a^b)| > \exp\left\{-\frac{s^3}{\log^3 s}(s+\log H)\log^{2+\epsilon}(s+\log H)\right\},$$

$$\left|P\left\{\frac{\log\alpha}{\log\beta}\right\}\right| > \exp\{-s^{\epsilon}(s+\log H)^{2+\epsilon}\},$$

if $H>H_0$.

On specializing the numbers η_i and α_k, (1) and (2) give, e.g., the following corollaries. (4) If α and $a\neq 0, \neq 1$ are algebraic and α is of higher than the third degree, then at least two of the numbers a^{α}, a^{α^2}, a^{α^3}, a^{α^4} are algebraically independent. If α is of the third degree, then already a^{α} and a^{α^2} are algebraically independent. [See the paper cited at the beginning of this review.] (5) If $a\neq 0, \neq 1$ is algebraic, and $\eta\neq 0$ is rational, then at least two of the numbers $a^{\exp(k\nu)}$, $k=1,2,3,4$, are algebraically independent. (6) If $\nu\neq 0$ is a rational number, then at least one of the numbers $\exp(e^{k\nu})$, $k=1,2,3$, is algebraically independent of e, thus transcendental; and at least one of the numbers $\exp(\pi^{k\nu+1})$, $k=1,2,3$, is transcendental. Theorem 3 is also of interest because, e.g., the Thue-Siegel theorem on algebraic numbers implies only the weaker inequality $|P\{(\log\alpha)/(\log\beta)\}| > e^{-\epsilon H}$.

The proofs of (1)–(3) are rather similar; they are based on a number of lemmas which are of interest in themselves. For instance, (1) is proved as follows. Assume that every two of the 11 numbers (a) are algebraically dependent. Then there exist two transcendental numbers ω and ω_1, where $\omega_1^{\nu}+c_1(\omega)\omega_1^{\nu-1}+\cdots+c_{\nu}(\omega)=0$ and $c_1(\omega), c_2(\omega), \cdots, c_{\nu}(\omega)$ are polynomials in ω with integral coefficients, such that the numbers (a) are of the form S_i/T_i $(i=1,\cdots,11)$, the S_i and T_i being polynomials in ω of arbitrary degree and in ω_1 of degree at most $\nu-1$ with integral coefficients. Denote by N a sufficiently large positive integer, and put $T=T_1T_2\cdots T_{11}$, and $p=[N^3\log^{-1}N]+1$, $p_1=[N^3/\log^{\frac{1}{2}}N]$, $\lambda=3+\tau$, $\mu=[11\lambda N^3\log^{-1}N]$. In the function

$$f(z)=\sum_{k_0=0}^{p-1}\sum_{k_1=0}^{N}\sum_{k_2=0}^{N}\sum_{k_3=0}^{N}A_{k_0k_1k_2k_3}z^{k_0}\exp\{(k_1\eta_1+k_2\eta_2+k_3\eta_3)z\}$$

let the A's be of the form

$$A_{k_0\cdots k_1}=\sum_{k_2=0}^{p_1}C_{k_0\cdots k_2}\omega^{k_2},$$

where the C's are integers not all zero. Then the numbers

$$f_{k_0k_1k_2}=T^{\nu}f(k_0+k_1\alpha_1+k_2\alpha_2),$$

where $0\leqq k_i\leqq\lambda N^3\log^{-1}N$, $i=0,1,2$, are polynomials in ω and ω_1 with integral coefficients, and at least one of them is not zero. Denote by $\lambda_2, \lambda_3, \cdots$ positive constants independent of N. By means of Dirichlet's principle, integral values of the C's may be chosen such that

$$|C_{k_0\cdots k_2}| < \exp(\lambda_2 N^2\log^{-1}N),$$

$f_{k_0k_1k_2}=0$, i.e., $f(k_0+k_1\alpha_1+k_2\alpha_2)=0$, for

$$0\leqq k_i\leqq[\lambda_2 N^2\log^{-1}N]=\mathfrak{K},$$

$i=0,1,2$. If Γ is the circle $|\zeta|=2N^{3/2}$, therefore,

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma}\prod_{k_0=0}^{\mathfrak{K}}\prod_{k_1=0}^{\mathfrak{K}}\prod_{k_2=0}^{\mathfrak{K}}\left\{\frac{z-k_0-k_1\alpha_1-k_2\alpha_2}{\zeta-k_0-k_1\alpha_1-k_2\alpha_2}\right\}\frac{f(\zeta)d\zeta}{\zeta-z},$$

and so

$$|f(z)| < \exp(-\lambda_3 N^3\log^{-1}N), \qquad |z|\leqq N^3,$$

whence

$$|f_{k_0k_1k_2}| < \exp(-\tfrac{1}{2}\lambda_3 N^3\log^{-1}N), \qquad 0\leqq k_i\leqq\lambda N^2\log^{-1}N;\ i=0,1,2.$$

Select now k_0, k_1, k_2 according to these inequalities such that $f_{k_0k_1k_2}\neq 0$. Then $f_{k_0k_1k_2}=P(\omega,\omega_1)$ is a polynomial in ω and ω_1 with integral coefficients. Let H be its height (i.e., the maximum of the absolute values of the coefficients), and n its degree in ω; it may be assumed of degree $\nu-1$ in ω_1. Then one shows that $n+\log H<\lambda_4 N^3\log^{-1}N$. Denote by $\omega_1, \omega_2, \cdots, \omega_{\nu}$ the conjugates of ω_1 over the field generated by ω. The norm $P_0(\omega)=\prod_{i=1}^{\nu}P(\omega,\omega_i)$ does not vanish and is a polynomial in ω with integral coefficients, say of degree n_0 and height H_0. One shows that

$$|P_0(\omega)| < \exp(-\lambda_5 N^3\log^{-1}N),$$
$$\max(n_0, \log H_0) < \lambda_6 N^3\log^{-1}N.$$

By a new test of transcendency given in the paper, ω must then be algebraic contrary to hypothesis. *K. Mahler.*

Fel'dman, N. I. The approximation of certain transcendental numbers. Doklady Akad. Nauk SSSR (N.S.) 66, 565–567 (1949). (Russian)

Fel'dman, N. I. On the measure of transcendency of the logarithms of algebraic numbers and elliptic constants. Uspehi Matem. Nauk (N.S.) 4, no. 1(29), 190 (1949). (Russian)

[In the second paper the author's initials are incorrectly given as N. M.] Let $P(x)$ denote a polynomial with integer coefficients, of order n and of height H. The following theorems are stated. (1) There exists a positive constant c_1 such that for any polynomial $P(x)$,

$$|P(\pi)| \geqq \exp(-c_1 n\cdot\max[n\log^2 n; \log H\cdot\log\log H]).$$

(2) Let $\alpha\neq 1$ denote an algebraic number. There exists a positive constant c_2 depending only on α and on the branch of $\log\alpha$ which has been chosen, such that

$$|P(\log\alpha)| \geqq \exp(-c_2 n^2(1+\log n)\max[n\log^2 n; \log H\cdot\log\log H]).$$

(3) Let $\wp(z)$ denote the elliptic function of Weierstrass with algebraic invariants g_2 and g_3, and let ω denote a period of $\wp(z)$. There exist a constant c_3 depending only on the values of g_2 and g_3 and on the period ω, such that

$$|P(\omega)| \geqq \exp(-c_3 n^4\max[n\log^5 n; \log H\cdot(\log\log H)^4]).$$

(4) Let us suppose again that the invariants g_2 and g_3 of $\wp(z)$ are algebraic numbers, and let us choose β in such a manner that $\wp(\beta)$ shall be an algebraic number. It follows that

$$|P(\beta)| \geqq \exp(-\max[\exp\{n^{2+\epsilon}+n^{(2+\epsilon)(1+\epsilon)+1}\}; \log H\cdot\exp\{n(\log\log H)^{1+\epsilon}\}]),$$

where κ and ϵ are positive numbers, and it is supposed that

$$\max(\log H, \exp m^{2+\epsilon}) \geqq C,$$

where the constant C depends on ϵ, κ, β, g_2 and g_3. Theorems 3 and 4 imply that ω and β are transcendental numbers, which was first proved by T. Schneider [Math. Ann. 113, 1–13 (1936)]. It is stated that these results are proved by means of the methods developed by A. O. Gelfond [Bull. Acad. Sci. URSS. Cl. Sci. Math. Nat. [Izvestiya Akad. Nauk SSSR] 1934, 623–634; Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] 1939, 509–518; these Rev. 1, 295; C. R. (Doklady) Acad. Sci. URSS (N.S.) 7 (1935 II), 177–182]. *A. Rényi* (Budapest).

Davenport, H., and Rogers, C. A. A note on the geometry of numbers. J. London Math. Soc. 24, 271–280 (1949).

Let $G(x_1, \cdots, x_r)$, $E(x_1, \cdots, x_r)$ and $H(x_{r+1}, \cdots, x_n)$ be distance functions (not necessarily convex), where $1 \leqq r \leqq n-1$. It is proved that the star bodies in n dimensions defined by $G^{r/n}H^{(n-r)/n} \leqq 1$ and $G^{r/n}(E+H)^{(n-r)/n} \leqq 1$ have the same critical determinant, possibly infinite. Seven direct applications are given; e.g., the critical determinant of the star body $|x|(x^2+|yz|) \leqq 1$ is 7 and of $|x|(x^2+y^2+z^2) \leqq 1$ is $\frac{1}{2}(23)^{\frac{1}{2}}$. These functions are closely related to $|x(x^2+yz)|$ and $|x(-x^2+y^2+z^2)|$. It is shown that the critical determinant of the former is at least 9.1 and that of the latter is finite but exceeds $\frac{1}{2}(23)^{\frac{1}{2}}$. *L. Tornheim.*

Davenport, H. Corrigendum: On a theorem of Tschebotareff. J. London Math. Soc. 24, 316 (1949).

The paper appeared in the same J. 21, 28–34 (1946); cf. these Rev. 8, 443; 10, 855.

Ledermann, Walter, and Mahler, Kurt. On lattice points in a convex decagon. Acta Math. 81, 319–351 (1 plate) (1949).

Among all symmetric convex decagons those are extreme for which the densest lattice packing covers the least area, i.e., for which V/Δ is least, V being the area of the decagon and Δ the determinant of a critical lattice. The extreme decagons are determined and are affinely equivalent but not equivalent to the regular decagon although the regular octagon is extreme among symmetric octagons. The method depends on the fact for convex bodies that 4Δ is the lower bound of the areas of the circumscribing hexagons; cf. K. Reinhardt [Abh. Math. Sem. Hamburg. Univ. 10, 216–230 (1934)] and K. Mahler [Nederl. Akad. Wetensch., Proc. 50, 692–703 = Indagationes Math. 9, 326–337 (1947); these Rev. 9, 10]. The proof is mostly analytic rather than geometric. The extreme decagon can be "smoothed" giving a smaller value for V/Δ, but not as small as that for the smoothed regular octagon [cf. op. cit.]; this supports the conjecture that the latter is extreme among all 2-dimensional convex bodies. *L. Tornheim* (Ann Arbor, Mich.).

Vinogradov, I. M. Improvement of the remainder term of some asymptotic formulas. Izvestiya Akad. Nauk SSSR. Ser. Mat. 13, 97–110 (1949). (Russian)

Let $h(-t)$ be the class number of the binary quadratic forms with determinant $-t$. The author improves on previous work of Gauss and himself to obtain, for each $\epsilon > 0$,

$$\sum_{t=1}^{N} h(-t) = 4\pi\{21\zeta(3)\}^{-1}N^{\frac{3}{2}} - (2/\pi^2)N + O(N^{\frac{3}{4}-\delta+\epsilon})$$

where $\delta = 1/405$. A previous paper of the author [Bull. Acad. Sci. Petrograd [Izvestiya Akad. Nauk] (6) 11, 1347–1378 (1917)] gave the error term $O(N^{5/6}\log^{\frac{1}{2}} N)$. The improved result depends on the sharpening of estimates of the number of lattice points in certain three-dimensional regions. This is accomplished by using results of the author and van der Corput on the estimation of exponential sums. The author states that his method enables him to show that the number of lattice points in the sphere of radius r and center at the origin is equal to its volume plus an error term not exceeding $r^{1.6-2\delta+\epsilon}$. *L. Schoenfeld* (Urbana, Ill.).

Petrović, Mihail. Elementary observations on the distribution of small prime numbers. Glas Srpske Akad. Nauka 189, 5–45 (1946). (Serbian)

The author considers the problem of finding the number of primes of the respective forms $6x+1$, $6x-1$ not exceeding a given small limit L. The practical method of actually counting the primes is not allowed. The author finds instead the number of composite numbers of the same form not exceeding L by counting approximately the number of lattice points under a certain hyperbola. The results are not too accurate. The number of primes $6x+1$ under 1000 comes out 69 rather than the correct 86. There are numerous examples. *D. H. Lehmer* (Berkeley, Calif.).

Ramaswami, V. On the number of positive integers less than x and free of prime divisors greater than x^c. Bull. Amer. Math. Soc. 55, 1122–1127 (1949).

The author proves that the number of positive integers less than x and free of prime divisors greater than x^c is equal to $x\phi(c)+O(x/\log x)$, where $\phi(c) > 0$ for $c > 0$ and the O-term is uniform for $c \geqq \delta > 0$; moreover $\phi(c)$ is a continuous function of c for all real c and an increasing function of c for $0 < c < 1$. Weaker versions of this theorem have been proved by Chowla and Vijayaraghavan [J. Indian Math. Soc. (N.S.) 11, 31–37 (1947); these Rev. 9, 332] and by Buhštab [Doklady Akad. Nauk SSSR (N.S.) 67, 5–8 (1949); these Rev. 11, 84], although Ramaswami seems actually to have obtained his result earlier than these other authors. (Buchstab considered more generally the positive integers of an arithmetic progression, which presents little additional difficulty.) It is a corollary to Ramaswami's proof that

$$\phi(b) - \phi(a) = \int_a^b \phi(u/(1-u))u^{-1}du$$

for $0 < a < b \leqq 1$. In a paper which has already appeared [Duke Math. J. 16, 99–109 (1949); these Rev. 10, 597], but which is logically a sequel to the present one, the author has studied the function $\phi(c)$ in great detail. [Cf. also the following review.] *P. T. Bateman* (Princeton, N. J.).

Ramaswami, V. Sequences satisfying $a_1 = 1$ and $(r+1)^{-1}a_r \leqq a_r - a_{r+1} \leqq r^{-1}a_{r-1}$ for $r > 1$. Math. Student 16 (1948), 31–33 (1949).

[In the original the subscript is omitted from a_1 in the title.] The function ϕ of the paper reviewed above is such that the sequence $a_r = \phi(1/r)$, $r = 1, 2, \cdots$, satisfies the properties mentioned in the title of the present paper. The author raises the question here whether these properties alone imply the known fact that $0 \leqq \phi(1/r) \leqq 1/\Gamma(r)$. The sequence $a_r = \lambda/r + (1-\lambda)/\Gamma(r)$, $0 < \lambda \leqq 1$, shows that this is not the case. However, the author proves the following weaker statement: If a sequence $\{a_r\}$ satisfies the conditions in the title, then $a_r \geqq 0$ for $r \geqq 2$, $ra_r \leqq (r+1)a_{r+1}$ for $r \geqq 2$, and either ra_r has a positive limit as r tends to infinity or else $a_r \leqq 1/\Gamma(r)$ for all r. *P. T. Bateman* (Princeton, N. J.).

Shapiro, Harold N. Powerfree integers represented by linear forms. Duke Math. J. 16, 601–607 (1949).

Consider the set $\mathfrak{U}_k$ of k-dimensional vectors with integer components. For two such vectors X_1, X_2 we write $X_1 \equiv X_2 \pmod{\Delta}$ if the congruence holds componentwise. A property P of vectors in $\mathfrak{U}_k$ is called periodic if there exists some integer $\Delta > 0$ such that $X_1 \equiv X_2 \pmod{\Delta}$ implies that either both X_1 and X_2 possess P or that neither does so. The main theorem proved by the author is then as follows. Let $f_1, \cdots, f_k$ be linear polynomials, P a periodic property in $\mathfrak{U}_k$, and $Q_P(N)$ the number of integers $x \leqq N$ for which the vector $(f_1(x), \cdots, f_k(x))$ possesses P and none of $f_1(x), \cdots, f_k(x)$ is divisible by the rth power of any prime. Then either $Q_P(N) = AN + O(N^{s/(r+1)+\epsilon})$, where $A > 0$. In addition, necessary and sufficient conditions are obtained in order that $Q_P(N)$ should not vanish identically, and a special case involving products of linear polynomials is considered. *L. Mirsky* (Sheffield).

***Čudakov, N. G. Vvedenie v Teoriyu L-Funkciĭ Dirihle. [Introduction to the Theory of Dirichlet's L-Functions].** OGIZ, Moscow-Leningrad, 1947. 203 pp.

This book, assuming only the elements of number theory and complex variable, deals with characters, Dirichlet L-functions and their zeros, primes in progressions and culminates in the three-prime theorem of Goldbach and Vinogradov. The theorems on the zeros of the L-functions are those of Page and Siegel; the proof of Siegel's theorem is based on the work of Heilbronn. For the most part, the proofs given are those which have been published previously by other authors. *L. Schoenfeld* (Urbana, Ill.).

Bellman, Richard. Wigert's approximate functional equation and the Riemann zeta-function. Duke Math. J. 16, 547–552 (1949).

Writing $V_k(z)$ for the Mellin transform of $\Gamma(s)^k$ and $W_k(z) = \sum_{n=1}^{\infty} d_{2k}(n) V_k(nz)$, the author proves an approximate functional relation connecting $W_k(z)$ and $W_k((2\pi)^{2k}/z)$. For $k = 1$ this reduces to Wigert's approximate functional equation for the Lambert series [Acta Math. 41, 197–218 (1917)]. This leads to a certain advance in the mean-value problem when the exponent is a multiple of 4, and gives a variant of existing proofs in the case of $|\zeta(\frac{1}{2}+it)|^4$. The above result combined with Plancherel's theorem reduces the mean-value problem for $|\zeta(\frac{1}{2}+it)|^{4k}$ to the consideration of

$$\int_{(2\pi)^k}^{\infty} |W_k(ue^{-i(\frac{1}{2}k\pi - t)})|^2 du,$$

which seems somewhat more tractable than the integral originally derived by Titchmarsh [The Zeta-Function of Riemann, Cambridge University Press, 1930, § 2.53]. It is remarked that this provides a reformulation of the Lindelöf hypothesis. *F. V. Atkinson* (Ibadan).

Atkinson, F. V. The mean-value of the Riemann zeta function. Acta Math. 81, 353–376 (1949).

Hardy and Littlewood first showed that $\zeta(\frac{1}{2}+it)$ possesses an asymptotic mean-value, $\int_0^T |\zeta(\frac{1}{2}+it)|^2 dt = T \log T + R(T)$, $R(T) = o(T \log T)$, as $T \to \infty$. Since then, the order of magnitude of the remainder term has been investigated successively by Littlewood, Ingham and Titchmarsh. The last-named showed that $R(T) = -T(1 + \log 2\pi - 2C) + O(T^{5/12})$. The problem has much in common with the Dirichlet divisor problem, as far as methods and results are concerned [see Atkinson, Quart. J. Math., Oxford Ser. 10, 122–128 (1939)]. This analogy is carried a step further by the author when he shows that the $O(T^{5/12})$ term possesses an expansion in terms of oscillating functions quite similar to that obtained by applying Voronoi's summation formula to $\sum_{n \leqq x} d(n)$. *R. Bellman* (Stanford University, Calif.).

Gupta, Hansraj. On a conjecture of Miller. J. Indian Math. Soc. (N.S.) 13, 85–90 (1949).

Let $\mu(n)$ be Möbius' inversion function and let $M_1(x) = \sum_{n \leqq x} \mu(n)$. The conjecture of Miller is that the function $M_2(x) = \sum_{n \leqq x} M_1(n)$ is negative for $x \geqq 3$. The paper presents a table of $M_2(25m)$ for $m = 1(1)800$, showing that the conjecture holds for $x \leqq 20000$. The average $A(n) = M_2(n)/n$ is also tabulated for $n = 100(100)20000$. According to an informal argument of Brun [C. R. Dixième Congrès Math. Scandinaves 1946, pp. 40–53 (1947); these Rev. 8, 446], $A(n) = -2 + 12n^{-1} + \cdots$. The table fails to bear this out. The author conjectures that $A(n) = O(\log n)$. *D. H. Lehmer* (Berkeley, Calif.).

Putter, Joseph. On a modular equation connected with partitions. Riveon Lematematika 3, 42–43, 53 (1949). (Hebrew. English summary)

The fact that $\eta(\tau)/\eta(\tau/11)$ and $\eta(11\tau)/\eta(\tau)$ are not connected by a modular equation of degree 11 is proved by considering what the sum of the roots of such an equation would be. Also the nonexistence of an identity of the form $\eta(24\tau)\sum p(11n+6)e^{(24n+13)2\pi i\tau} = R(\eta(11.24\tau)/\eta(24\tau))$, R a rational function, is verified. *H. S. Zuckerman.*

Estermann, T. On Waring's problem: A simple proof of a theorem of Hua. Sci. Rep. Nat. Tsing Hua Univ. Ser. A. 5, 226–239 (1948).

This paper gives a self-contained proof of a theorem of Hua [Quart. J. Math., Oxford Ser. 9, 199–202 (1938)] that the Hardy-Littlewood asymptotic formula for the number of representations of an integer as the sum of s kth powers is valid if $s \geqq 2k+1$. For large k, this result is, of course, inferior to that obtained by the methods of Vinogradov. *L. Schoenfeld* (Urbana, Ill.).

Selberg, Sigmund. A survey of some recent results in additive number theory. Mat. Tidsskr. A. 1949, 1–15 (1949). (Norwegian)

This survey is concerned with the density of sums of sets of integers.

ANALYSIS

Hanai, Sitiro. A note on generalized convex functions. Proc. Japan Acad. 21 (1945), 378–381 (1949).

Let $\{F(x)\}$ be a family of functions continuous for $a \leqq x \leqq b$ and such that there is a unique member of the family which, at arbitrary x_1, x_2 satisfying $a \leqq x_1 < x_2 \leqq b$, takes on arbitrary values y_1, y_2. For a given function $f(x)$, and for $a \leqq x_1 < x_2 \leqq b$, let $F_{12}(x)$ be the member of $\{F(x)\}$ satisfying $F_{12}(x_i) = f(x_i)$, $i = 1, 2$. Then $f(x)$ is said to be convex relative to $\{F(x)\}$ provided $f(x) \leqq F_{12}(x)$ for all x_1, x_2, x with $a \leqq x_1 < x < x_2 \leqq b$. It is shown that the set of all functions $f(x)$ convex relative to a given $\{F(x)\}$ forms a conditionally complete lattice. An example is given to show that the above set of functions

$f(x)$ does not necessarily form a semi-vector lattice. However, it is proved that the set $\{f(x)\}$ does form such a lattice provided the family $\{F(x)\}$ satisfies the following additional conditions. Let $F(x)$, $F^*(x)$ be the members of $\{F(x)\}$ whose graphs pass through arbitrary points (x_1, y_1), (x_2, y_2) and (x_1, y_1^*), (x_2, y_2^*), respectively; let $F_\lambda(x)$, $\lambda>0$, be the member whose graph passes through $(x_1, \lambda y_1)$, $(x_2, \lambda y_2)$; and let $F^{**}(x)$ be the member whose graph passes through $(x_1, y_1+y_1^*)$, $(x_2, y_2+y_2^*)$. Then for $a\leqq x_1<x<x_2\leqq b$ we have $F_\lambda(x)\geqq\lambda F(x)$ [which, we note, is equivalent to $F_\lambda(x)=\lambda F(x)$] and $F^{**}(x)\geqq F(x)+F^*(x)$. In particular, the set of all Phragmén-Lindelöf functions $h(x)$ forms a semi-vector lattice, and, as was previously known, so does the set of all ordinary convex functions. *E. F. Beckenbach.*

Chaves Peixoto, Marília. On the inequalities $y'''\geqq G(x, y, y', y'')$.
Anais Acad. Brasil. Ci. **21**, 205–218 (1949).

M. Matos Peixoto has shown [Bull. Amer. Math. Soc. **55**, 563–572 (1949); these Rev. **10**, 686] that, if the family $\{f(x)\}$ [see the preceding review] consists of the solutions of a differential equation of the form $y''=G(x, y, y')$, then under suitable conditions on $\{F(x)\}$ a function $f(x)$ of class C'' is convex relative to $\{F(x)\}$ if and only if $f''\geqq G(x, y, y')$. The family $\{F(x)\}$ might be replaced by a family $\{\Phi(x)\}$ of continuous functions such that there is a unique member of the family which, at arbitrary $x_1, \cdots, x_n$, satisfying $a\leqq x_1<\cdots<x_n\leqq b$, takes on arbitrary values $y_1, \cdots, y_n$. With notation analogous to that used in the preceding review, a given function $\varphi(x)$ is said to be convex relative to $\{\Phi(x)\}$ provided $\varphi(x)$ is alternately greater than or equal to and less than or equal to $\Phi_{1,\cdots,n}(x)$ in the intervals (x_j, x_{j+i}), with $\varphi(x)\leqq\Phi_{1,\cdots,n}(x)$ in (x_{n-1}, x_n). In the present paper the above differential characterization is extended to the case $n=3$, with the differential equation and inequality replaced by $y'''=H(x, y, y', y'')$ and $\varphi'''\geqq H(x, \varphi, \varphi', \varphi'')$, respectively. *E. F. Beckenbach* (Los Angeles, Calif.).

Jecklin, H. Quasiarithmetische Mittelwerte. Elemente der Math. **4**, 112–115 (1949).

After having given the usual definition of quasiarithmetic means:

$$m_n(x_1, \cdots, x_n)=\phi\left(\sum_{i=1}^n k_i f(x_i)\Big/\sum_{i=1}^n k_i\right)$$

(the k_i's are positive; $F(x)$ is continuous and strictly monotonic, having the inverse function ϕ) and after having mentioned their fundamental properties

$$(\min(x_1, \cdots, x_n)\leqq m_n(x_1, \cdots, x_n)\leqq\max(x_1, \cdots, x_n);$$

m_n remains unchanged if we substitute for $f(x)$ an $af(x)+b$ or for a number h of x_i's their partial quasiarithmetic mean m_h), the author announces a simple and suggestive proof of Jensen's inequality:

$$f\left(\sum_{i=1}^n k_i x_i\Big/\sum_{i=1}^n k_i\right)=\sum_{i=1}^n k_i f(x_i)\Big/\sum_{i=1}^n k_i$$

if $f(x)$ is convex (this inequality makes it possible to compare quasiarithmetic means having monotonic and convex or concave f functions with the arithmetic mean). The author supposes only $f[(x_1+x_2)/2]\leqq[f(x_1)+f(x_2)]/2$, but he uses in his proof geometric facts equivalent to

$$f[(k_1x_1+k_2x_2)/(k_1+k_2)]\leqq[k_1f(x_1)+k_2f(x_2)]/(k_1+k_2).$$

He does not say why his proof is simpler and more suggestive than the usual one. [Actually his proof is equivalent to the usual one: see, e.g., Hardy, Littlewood, and Pólya, Inequalities, Cambridge University Press, 1934, p. 73 (3.7.1).] *J. Aczél* (Szeged).

Jecklin, H. Quasiarithmetische Mittelwerte. II. Elemente der Math. **4**, 128–133 (1949).

[Cf. the preceding review.] The root-mean-power, the geometric, the exponential, and the sine means are given as examples for quasiarithmetic means and their comparison with the arithmetic mean. The author gives a geometric construction of the quasiarithmetic mean constructed with a given function. The author describes a proof of $\sum_1^n x_i^2/\sum_1^n x_i\geqq(\sum_1^n x_i^2/n)^{\frac{1}{2}}$ and further proofs of $(\sum_1^n x_i^2/n)^{\frac{1}{2}}\geqq\sum_1^n x_i/n\geqq(\prod_1^n x_i)^{1/n}\geqq n/[\sum_1^n(1/x_i)]$.
J. Aczél (Szeged).

Zappa, G. Osservazioni sopra le medie combinatorie. Metron **14**, 31–53 (1940).

The author defines different mean values formed by combining the root-mean-power with the combinatorial means, in which repetitions are taken into consideration in different manners. He proves theorems on the comparability of these means. In passing from discrete arguments to continuous ones some of these means will coincide. There are also theorems on the limit of these means for arguments forming arithmetic, geometric, etc. sequences with fixed extreme terms while the number of terms tends to infinity.
J. Aczél (Szeged).

Pizzetti, Ernesto. Medie ascendenti e medie discendenti. Metron **14**, 55–66 (1940).

The author examines properties of arithmetic means with weights specially chosen. He proves theorems such as the following. The arithmetic mean of a nondecreasing sequence a_m $(m=0, 1, 2, \cdots, n)$ with the weights $\binom{n-m+k-1}{k-1}$ $(m=0, 1, 2, \cdots, n)$ increases with n and decreases with k, having the limit a_0. He gives the analogous theorem for nonincreasing sequences and for mean values which can be transformed into the arithmetic mean (such as the geometric and the harmonic mean). The special cases where a_m is an arithmetic sequence or a geometric sequence are discussed in detail. *J. Aczél* (Szeged).

Obrechkoff, N. Sur quelques inégalités pour les dérivées des fonctions. C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 1, 1–4 (1949).

The following theorem is proved by induction. Let $f(x)$ be real and have an nth derivative which is nonnegative for $x>x_0$. If there is a sequence $y_k\uparrow\infty$ such that $f(y_k)/y_k^m=0$ for some m, $0\leqq m<n$, then $(-1)^{n-m}f^{(m)}(x)\geqq 0$ for $x>x_0$. Analogous results are given for differences of functions and of sequences. *R. P. Boas, Jr.* (Providence, R. I.).

Obrechkoff, N. Sur une formule pour les différences divisées et sur les limites de fonctions et de leurs dérivées. C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 1, 5–8 (1949).

The author proves several theorems which contain results of several authors as special cases. The following is typical. Let $f(x)$ be real and have an nth derivative for $x>a$, satisfying $f^{(n)}(x)>-Mx^{k-n}$, $M>0$; let $\lim_{x\to\infty}x^{-k}f(x)=A$. Then for $1\leqq p\leqq n-1$, we have $x^{p-k}f^{(p)}(x)\to Ak(k-1)\cdots(k-p+1)$, $x\to\infty$. The proofs depend on a mean value theorem involving divided differences. *R. P. Boas, Jr.*

Ilieff, Ljubomir. Ganze Funktionen mit lauter reellen Nullstellen. C. R. Acad. Bulgare Sci. Math. Nat. **2**, no. 1, 17–20 (1949).

A short proof is given of the following theorem. Let $f(t)$ and $\psi(t)$ be nonnegative and integrable and let $F(z)=\int_0^1 f(t)\sin zt\,dt$ have only real (or real and simple) zeros. Let $\omega(t)=\int_t^1\psi(t)dt$, $x=\omega(t)$, $t=\tilde{\omega}(x)$ the inverse of $\omega(t)$; $f(t)=\psi(t)f_0(t)$, $f_0(t)=\varphi_0(x)$, $f_1(t)=\int_0^x\varphi_0(u)du$. Then $F_1(z)=\int_0^1 f_1(t)\cos zt\,dt$ has only real (or real and simple) zeros. This enables the author to derive various classes of decreasing functions $f_1(t)$ for which $F_1(z)$ has only real zeros by using known classes of increasing $f(t)$ for which $F(z)$ has only real zeros. *R. P. Boas, Jr.* (Providence, R. I.).

Salem, R. Convexity theorems. Bull. Amer. Math. Soc. **55**, 851–860 (1949).

This address gives a survey of a convexity theorem due to M. Riesz and its extensions. In the first part the author states and illustrates the Riesz theorem, with its application to the Hausdorff-Young theorem and others, following the papers of M. Riesz [Acta Math. **49**, 465–497 (1926)] and of A. Zygmund [Prace Mat.-Fiz. **44**, 91–107 (1937); Trigonometrical series, Warsaw, 1935]. The second part is devoted to Thorin's generalization of Riesz's theorem [Kungl. Fysiografiska Sällskapets i Lund Förhandlingar [Proc. Roy. Physiog. Soc. Lund] **8**, no. 14 (1939)]. The author sketches his short proof of this theorem [Colloquium Math. **1**, 6–8 (1947); these Rev. **9**, 274; note that in this paper it should have been stated that $\log|f|$, not $|f|$, is subharmonic]. As an application the author states that a deep theorem due to Littlewood and Paley [Proc. London Math. Soc. (2) **43**, 105–126 (1937)] can be easily derived from Thorin's theorem. The author concludes by some further applications.

S. Izumi (Sendai).

Geronimus, Ya. L. On the degree of precision of quadrature formulas. Doklady Akad. Nauk SSSR (N.S.) **68**, 437–440 (1949). (Russian)

Consider the quadrature formula

$$\int_{-1}^{1} f(x)d\sigma(x)=\sum_{i=1}^{n}\lambda_i^{(n)}f(y_i^{(n)}),\quad \{\lambda_i^{(n)}\}_1^n>0;$$

$$-1\leqq y_1^{(n)}<\cdots<y_n^{(n)}\leqq 1, \tag{1}$$

which is assumed to be valid for all polynomials of degree not exceeding M_n. The author obtains certain upper bounds for the number M_n in terms of the properties of $\sigma(x)$ and the interval of the subdivision (1) nearest to -1. Let $P_n(x)$, $n=0,\cdots,\infty$, be a system of polynomials orthogonal to $d\sigma(x)$ and put

$$P_m(x)=\prod_{k=1}^{m}(x-x_k^{(m)}),\quad x_k^{(m)}=-\cos\theta_k^{(m)},$$

$$0<\theta_1^{(m)}<\theta_2^{(m)}<\cdots<\theta_m^{(m)}<\pi.$$

Write $\tau(\theta)=\sigma(-\cos\theta)$ and let $\omega(m)$ be a decreasing function of m such that $\theta_2^{(m)}\leqq\omega(m)$. Let z_n denote the distance from -1 to the point of (1) nearest to -1 but not identical with -1. Then writing ω^{-1} for the inverse of ω, the author shows that $M_n<2\omega^{-1}(\sqrt{(2z_n)})$ and if $y_1^{(n)}>-1$, $y_n^{(n)}<1$, then $M_n<2\omega^{-1}(\tau^{-1}(\lambda_1^{(n)}))$. The author then determines a form for $\omega(m)$ which enables him to give further estimates for M_n. *A. C. Offord* (London).

Čakalov, Lyubomir N. On a general quadrature formula. Doklady Akad. Nauk SSSR (N.S.) **68**, 233–236 (1949). (Russian)

Let $a_1,\cdots,a_m$ denote m distinct real numbers and $r_1,\cdots,r_m$, m corresponding nonnegative integers. Let $M+1=\sum_{k=1}^{m}(r_k+1)$ and let $\psi(x)$ be some function of bounded variation in $-\infty<x<\infty$ for which the Stieltjes integrals

$$\int_{-\infty}^{\infty}x^l d\psi(x),\qquad l=1,\cdots,M,$$

converge. By a general quadrature formula we mean an expression of the form

$$\int_{-\infty}^{\infty}f(x)d\psi(x)=\sum_{k=1}^{\infty}\sum_{\lambda=0}^{r_k}A_{k\lambda}f^{(\lambda)}(a_k), \tag{1}$$

the coefficients of which do not depend on $f(x)$. Let

$$P(x)=(x-a_1)^{r_1+1}\cdots(x-a_m)^{r_m+1}$$

and write

$$R(z)=\{P(z)\}^{-1}\int_{-\infty}^{\infty}(z-x)^{-1}[P(z)-P(x)]d\psi;$$

then the formula (1) will be valid for all polynomials of degree not exceeding M, when and only when the coefficients $A_{k\lambda}$ are given by the formula

$$R(z)=\sum_{k=1}^{\infty}\sum_{\lambda=0}^{r_k}(z-a_k)^{-\lambda-1}\lambda!A_{k\lambda}.$$

A. C. Offord (London).

Tchakaloff, L. (Čakalov, L.). Sur un problème de D. Pompéiu. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **40**, 1–14 (1944). (Bulgarian. French summary)

Let Ω be a domain in the (x, y)-plane, and let G be a group of Euclidean transformations. A general problem is the determination of the set of functions $f(x, y)$ continuous in the finite plane and satisfying

$$\int\int_{\Omega'}f(x, y)dxdy=0 \tag{1}$$

for all Ω' obtained from Ω by transformations of G. By investigating the zeros of the entire function

$$F(\lambda,\mu)\equiv\int\int_{\Omega}e^{\lambda x+\mu y}dxdy,$$

the author demonstrates the existence of nontrivial real solutions of (1) for arbitrary Ω provided G is the group of translations. It is pointed out that the solutions obtained include solutions of the previously solved problem in which Ω is a circle and G is the group of all Euclidean transformations. The method applies equally well to an analogous three-dimensional problem. *E. F. Beckenbach.*

Christov, Chr. Über eine Integraleigenschaft der Funktionen von zwei Argumenten. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **39**, 395–408 (1943). (Bulgarian. German summary)

This is an earlier presentation of the author's demonstration [Mathematica, Timişoara **23**, 103–107 (1948); these Rev. **10**, 20] that the only solution of the problem of the preceding review, in which now Ω is a square and G is the group of all Euclidean transformations, is the trivial solution $f(x, y)\equiv 0$. *E. F. Beckenbach* (Los Angeles, Calif.).

Delange, Hubert. On two theorems of S. Verblunsky. Proc. Cambridge Philos. Soc. **46**, 57–66 (1950).

Of the two theorems in question the more important one states that corresponding to any monotone function $\sigma(x)$ with $\sigma(+\infty)-\sigma(-\infty)<\infty$ there exists a function $f(x)$ with $0\leqq f(x)\leqq 1$, such that

$$\exp\left(\int_{-\infty}^{\infty}(\zeta-x)^{-1}f(x)dx\right)=1+\int_{-\infty}^{\infty}(\zeta-x)^{-1}d\sigma(x)$$

for ζ nonreal, and vice versa, and it expresses a connection between two moment problems. Correspondingly, Verblunsky employed for the proof the theory of positive harmonic functions in the half-plane, whereas the author succeeds in using directly partial fraction decomposition of rational functions, although on the face of it this would be a somewhat more remote approach. *S. Bochner.*

Scott, W. T. The corresponding continued fraction of a *J*-fraction. Ann. of Math. (2) **51**, 56–67 (1950).

The moment problem $c_p=\int_0^\infty u^p d\phi(u)$, $p=0, 1, 2, \cdots$ (Stieltjes), has a bounded nondecreasing solution $\phi(u)$ having infinitely many points of increase if, and only if, the formal power series $P(1/z)=\sum(-1)^p c_p/z^{p+1}$ has a continued fraction expansion of a certain type ("*S*-fraction") in which the coefficients satisfy certain conditions. The analogous problem in which the range of integration is extended to the whole real axis (Hamburger) has a solution if, and only if, $P(1/z)$ has a continued fraction expansion of a different type ("*J*-fraction"), with suitable coefficients; $P(1/z)$ does not necessarily have an *S*-fraction expansion. In this paper, Scott finds a generalization of the *S*-fraction, namely

$$\frac{a_0}{z^{\delta_1}}-\frac{a_1}{z^{\delta_2}}-\frac{a_2}{z^{\delta_3}}-\cdots \tag{1}$$

in which $\delta_1=1$, δ_p is 0 or 1, $\delta_p+\delta_{p+1}>0$, $p=1, 2, 3, \cdots$, and proves that the moment problem for the interval $[-\infty, \infty]$ has a solution if, and only if, $P(1/z)$ has a continued fraction expansion of this form in which $a_0>0$, $a_{p+1}>0$ if $\delta_{p+1}=\delta_{p+2}=1$, $a_{p+1}a_{p+2}>0$ if $\delta_{p+1}=1$, $\delta_{p+2}=0$. For the particular case of the moment problem for $[0, \infty]$, the conditions are $a_0>0$, $a_p<0$, $\delta_{2p-1}=1$, $\delta_{2p}=0$. Conditions for uniqueness of the solution are expressed in terms of the continued fraction. The notion of positive definiteness is extended to (1), two invariability theorems are proved, and an application is made to the problem of finding necessary conditions for convergence of a continued fraction. *H. S. Wall.*

Ghizzetti, Aldo. Sul problema dei momenti. Univ. e Politecnico Torino. Rend. Sem. Mat. **8**, 93–107 (1949). Expository lecture.

R. P. Boas, Jr. (Providence, R. I.).

Calculus

***Duschek, Adalbert. Vorlesungen über höhere Mathematik. Band I. Integration und Differentiation der Funktionen einer Veränderlichen. Anwendungen. Numerische Methoden. Algebraische Gleichungen. Grundzüge der Wahrscheinlichkeitsrechnung.** Springer-Verlag, Wien, 1949. x+395 pp. $7.80; bound, $8.70.

Zahlen und Zahlenfolgen. Der Funktionsbegriff. Das Integral und die Ableitung. Die elementaren transzendenten Funktionen. Ergänzungen zur Differential- und Integralrechnung. Polynome, algebraische Gleichungen und rationale Funktionen. *Table of contents.*

***Fubini, Guido, e Albenga, Giuseppe. La Matematica dell'Ingegnere e le sue Applicazioni.** Vol. 1. Nicola Zanichelli, Bologna, 1949. viii+498 pp. 4000 lire.

Teoremi varii di algebra elementare. Determinanti ed equazioni di primo grado. Unità di misura. Nozioni fondamentali su segmenti, angoli, aree orientate e sui vettori. Altere nozioni fondamentali. Coordinate cartesiane e prime applicazioni. Teoria di Gauss dei sistemi ottici centrati. Elementi di geometria descrittiva. Vettori applicati; momenti. Coordinate polari e numeri complessi. Elementi di geometria analitica piana. Limiti, continuità, derivate. Gli elementi del calcolo differenziale. Massimi, minimi, flessi. Gli elementi del calcolo integrale. Alcune prime applicazioni alla fisica delle nozioni di derivata e di integrale. L'ellisse. Coniche e curve piane di speciale importanza. Luoghi nel spazio. *Table of contents.*

Obrechkoff, N. Sur une nouvelle formule du calcul différentiel. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **39**, 381–393 (1943). (Bulgarian. French summary)

The author has given a generalization of Taylor's formula involving the Cesàro means of integral order [see Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1940**, no. 4; G. Kowalewski, Deutsche Math. **6**, 349–351 (1942); these Rev. **2**, 284; **4**, 273]. Written as a formula for $f(a+h)$, it involves the kth mean of the terms through h^n of the Taylor series, k derivatives evaluated at $a+h$, and a remainder R_n expressed as an integral involving $f^{(n+k+1)}$. The author gives a new derivation which allows him to replace the integral form for R_n by one involving $f^{(n+k+1)}(a+\theta h)$ and to prove the formula under weaker hypotheses. He then establishes the corresponding formula for the complex domain, with the remainder given by a contour integral, and shows that, for $f(z)=(1-z)^{-1}$, $R_n\to 0$ for $|z|^2<4|z-1|$ and $R_n\to\infty$ for $|z|^2>4|z-1|$. [Cf. also Hummel and Seebeck, Amer. Math. Monthly **56**, 243–247 (1949); these Rev. **10**, 516, and the reference given in the review.] *R. P. Boas, Jr.*

Bruwier, L. Sur une propriété des fonctions continues. Mathesis **58**, 21–23 (1949).

Given: $g(t)$ is continuous, $t_0\leqq t\leqq t_1$; $g(t_0)=a\neq g(t_1)=b$. For convenience in statement, suppose $a<b$. It is proved by obvious means that there exists a segment $T_0\leqq t\leqq T_1$ with $a<T_0<T_1<b$ such that $g(T_0)=a$, $g(T_1)=b$, and $a<g(t)<b$ when $T_0<t<T_1$. The number of such segments is finite. An application: Given $f(x)$ continuous, $a\leqq x\leqq b$, and $g(t)$ continuous together with $g'(t)$, $t_0\leqq t\leqq t_1$, with $g(t_0)=a$ and $g(t_1)=b$. Then, taking T_0 and T_1 as above, $\int_a^b f(x)dx=\int_{T_0}^{T_1}f[g(t)]g'(t)dt$. [Compare de la Vallée Poussin, Cours d'Analyse Infinitésimale, v. 1, 4th ed., Gauthier-Villars, Paris, 1921, p. 214.] *A. B. Brown.*

Petrovitch, Michel. Sur une classe d'intégrales définies dépendant d'un paramètre. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **5**, 93–107 (1939).

The integrals are of the form $U(x)=\int_a^b(u+xv)e^{xt}dt$, with u and v each functions of t. *P. Franklin.*

Castoldi, Luigi. Attorno a un limite notevole. Boll. Un. Mat. Ital. (3) **4**, 128–129 (1949).

Evaluation of the limit of $\int_0^{\frac{1}{2}\pi}\beta^n e^{-\beta\cos t}d\theta$ as $\beta\to+\infty$. The author does not seem to be aware of the fact that the

integral is well known from the standard proof of Jordan's lemma [cf., for instance, E. T. Copson, An Introduction to the Theory of Functions of a Complex Variable, Oxford, 1935, § 6.52]. For $\alpha \neq 0$ the author's conclusion checks with the known answer; for $\alpha = 0$ there is a contradiction.

A. Erdélyi (Pasadena, Calif.).

Ghosh, P. K. On Abel-convergent integrals and their application to mathematical physics. Bull. Calcutta Math. Soc. **41**, 143–152 (1949).

The author justifies various rules of integral calculus for divergent integrals which are Abel summable, and applies them to give rigorous discussions of some formal calculations arising in scattering problems in quantum mechanics.

R. P. Boas, Jr. (Providence, R. I.).

Truesdell, Clifford. Deux formes de la transformation de Green. C. R. Acad. Sci. Paris **229**, 1199–1200 (1949).

The author develops two forms of Green's theorem which appear to have application in mechanics and physics. The first of the author's results leads to a formula due to H. Lamb and J. J. Thomson for the energy of an incompressible fluid as well as to a formula of H. Poincaré. The second form of Green's theorem leads to a generalization of some theorems of A. Foppl and R. Berker.

N. Coburn (Ann Arbor, Mich.).

Theory of Sets, Theory of Functions of Real Variables

Rado, R. Axiomatic treatment of rank in infinite sets. Canadian J. Math. **1**, 337–343 (1949).

Concepts of rank, independence and basis are developed for sets of arbitrary cardinal number. This is an extension of similar ideas, applicable to finite sets, due to H. Whitney [Amer. J. Math. **57**, 509–533 (1935)]. There are considered: an arbitrary set M, the class of finite subsets $A \subset M$, and a nonnegative integer-valued function $r(A)$ such that (1) $r(\theta)=0$, θ being empty; (2) $r(A) \leqq r(A+\{x\}) \leqq r(A)+1$; (3) if $r(A)=r(A+\{x\})=r(A+\{y\})$, then $r(A+\{x,y\})=r(A)$. Such a function r is called a rank function in M. A set L ($\subset M$) is called independent if, for every finite set $A \subset L$, $r(A)$ is the cardinal number of A. A set L is called a basis for M if it is independent and is not a proper subset of any independent subset of M. The principal results are as follows. (1) Every set in which there is a rank function has a basis. (2) All bases for M have the same cardinal number.

L. W. Cohen (Flushing, N. Y.).

Tsuchikura, Tamotsu. Quelques propositions équivalentes à l'hypothèse du continu. Tôhoku Math. J. (2) **1**, 69–76 (1949).

The author proves that the continuum hypothesis is equivalent to each of three propositions that are modifications of the propositions C_9, C_{10}, C_{11} of W. Sierpiński [Hypothèse du Continu, Warszawa-Lwów, 1934, pp. 52–53].

A. Rosenthal (Lafayette, Ind.).

Iseki, Kiyosi. Sur les ensembles singulières. I. Une proposition équivalente à l'hypothèse du continu. J. Osaka Inst. Sci. Tech. Part I. **1**, 3–4 (1949).

The author proves that the following proposition is equivalent to the continuum hypothesis: the interval $[0, 1]$ is the sum of disjoint denumerable sets such that every sum of nondenumerably many of these sets contains at least one point of each nondenumerable G_δ-set of measure zero. [The proof contains quite a few misprints.] Then the following problem of Ruziewicz is considered: if the interval $[0, 1]$ is decomposed into disjoint pairs of points, is it always possible, by taking certain of these pairs, to form a set of any given measure α (with $0<\alpha<1$)? The author contends that, on the basis of the continuum hypothesis, a negative answer to Ruziewicz's problem follows from his result. The reviewer fails to see this.

A. Rosenthal (Lafayette, Ind.).

Dolcher, Mario. Nozione generale di struttura per un insieme. Rend. Sem. Mat. Univ. Padova **18**, 265–291 (1949).

L'auteur propose de considérer deux structures sur un même ensemble comme identiques lorsque leurs groupes d'automorphismes sont conjugués dans le groupe de toutes les permutations de l'ensemble. Cela l'amène par exemple à identifier, sur un ensemble à quatre éléments, la structure de groupe cyclique et la structure d'ordre définie par $a<b$, $b<c$, $b<d$, conclusion que peu de mathématiciens considéreront comme satisfaisante: c'est oublier en effet que ce qui fait l'intérêt de la théorie d'une espèce de structures (telles les structures de groupe, d'espace topologique, d'ensemble ordonné) c'est qu'on peut formuler les axiomes de ces structures pour des ensembles de base non explicités. Il n'y a donc lieu d'identifier deux espèces de structures que lorsqu'on peut établir une correspondance biunivoque entre elles, qui ne dépende pas de l'ensemble sur lequel on les considère.

J. Dieudonné (Nancy).

Théodoresco, N. L'application des ensembles et les conditions de biunivocité ponctuelle. Bull. Math. Soc. Roumaine Sci. **48**, 145–157 (1947).

[The article is paged 129–141, but the pagination is corrected at the end of the issue.] Let A and B be sets, and let 2^A and 2^B be the classes of all subsets of A and B, respectively. A one-to-one exhaustive mapping of A in B generates in an obvious way a (1) one-to-one, (2) union-preserving, (3) exhaustive mapping T of 2^A in 2^B. It is here shown that also the converse is true. Each of the conditions (1), (2), (3) is then dissected into two and the resulting system of axioms on T shown independent.

R. Arens.

Hadwiger, H. Die Multikongruenz und der Satz von Banach und Tarski. Abh. Math. Sem. Univ. Hamburg **16**, 48–53 (1949).

Two point sets A and B of k-dimensional Euclidean space are multicongruent (of degree n) (in notation, $A \simeq_n B$) if they admit decompositions $A=A_1+\cdots+A_n$, $B=B_1+\cdots+B_n$ into disjoint subsets such that $A_i \simeq B_i$ ($i=1, \cdots, n$), i.e., A_i and B_i are congruent. In addition to the properties of being reflexive, symmetric and transitive, multicongruence has the following two properties. (1) If $A_1 \varepsilon A$, $A \simeq_n B$, then there exists a subset $B_1 \varepsilon B$ such that $A_1 \simeq_n B_1$. (2) If $C \varepsilon B \varepsilon A$, and $A \simeq_n C$, then $A \simeq_{n+1} B$. While (1) is obvious, (2) is not. The latter was first proved by Sierpiński [Fund. Math. **33**, 229–234 (1945); these Rev. **8**, 17] by means of a theorem of Banach on biunique transformations [Fund. Math. **6**, 236–239 (1924)]. This paper gives a brief direct proof of property (2). The remainder of the paper confines itself to Euclidean 3-space S (though extension to k-space (for $k>3$, but not $k<3$) is easy) and covers the congruence theorem of Banach and Tarski [Fund. Math. **6**, 244–277 (1924)], namely: if N and M are bounded sets of S, each containing

inner points, then there exists a positive integer n such that $N \simeq_n M$. The author gives a much abbreviated proof of this theorem retaining the main considerations of the original.

H. Blumberg (Columbus, Ohio).

Sierpiński, Wacław. Sur l'operation $\overline{\lim}_{x \to +\infty} \Phi(x, y)$. Fund. Math. **36**, 51–55 (1949).

This note supplies proofs for theorems previously announced by the author [Pont. Acad. Sci. Acta **4**, 203–204 (1940); these Rev. **2**, 256]. It is shown that there exist $\Phi(x, y, z)$ of Baire class 2 such that the statement "$f(x) = \liminf_{x \to +\infty} \limsup_{y \to +\infty} \Phi(x, y, z)$ is measurable" is not decidable.

L. W. Cohen (Flushing, N. Y.).

Koksma, J. F., and Salem, R. Uniform distribution and Lebesgue integration. Acta Sci. Math. Szeged. **12**, Leopoldo Fejér et Frederico Riesz LXX annos natis dedicatus, Pars B, 87–96 (1950) = Math. Centrum Amsterdam. Rapport **ZW 1949-004**, 9 pp. (1949).

The following is the main result of the paper. Let $f(x)$ be real-valued, of period 1, of mean value zero and of the class L^2. Suppose that the Fourier coefficients c_n of f satisfy the condition $\sum_1^n |c_k|^2 = O(\log n)^{-\alpha}$, where $\alpha > 1$. Let $u_1, u_2, \cdots$ be a sequence uniformly distributed mod 1 and satisfying for k, M, N integral and positive the condition $|\sum_{n=M+1}^{M+N} \exp(2\pi i k u_n)| \leqq \Delta k^\rho N^\sigma (M+N)^\tau$, where Δ, ρ, σ, τ are constants such that $\sigma + \tau < 1$, $\tau < \frac{1}{2}$. Then

$$\lim N^{-1}\{f(x+u_1) + f(x+u_2) + \cdots + f(x+u_N)\} = 0$$

for almost every x. An example, due to Erdös and reproduced in the paper, shows that, from a certain point of view, some conditions upon the order of the coefficients c_n seem unavoidable.

A. Zygmund (Chicago, Ill.).

Zwirner, Giuseppe. Condizioni sufficienti per la complanarità dei punti di un arco di curva. Rend. Sem. Mat. Univ. Padova **18**, 197–202 (1949).

Given a curve γ in (x, y, z)-space, continuous image of a segment $a < t < b$ of the t axis, suppose that at each point the right semi-tangent is parallel to the (x, y)-plane. (By right semi-tangent is meant the limiting position of the secant from a point for which $t = t_0$ through a point for which $t = t_0 + h$, $h > 0$, as h approaches zero.) The author states that z is constant along γ. However, the proof seems to the reviewer to be incomplete. On page 201, line 11, "contenente punti di E" seems to be written without justification.

A. B. Brown (Flushing, N. Y.).

Pagni, Mauro. Un'osservazione sulle densità degli insiemi. Rend. Sem. Mat. Univ. Padova **18**, 228–230 (1949).

Let E be a bounded measurable set of the Euclidean space $R_n \equiv (x_1, \cdots, x_n)$. Through every point $P(\xi_1, \cdots, \xi_n)$ of E consider the hyperplane $R_r(P)$ $(r < n)$, determined by $x_1 = \xi_1, \cdots, x_{n-r} = \xi_{n-r}$. If its intersection with E is measurable then we form the quotient

$$m_r(E \cdot q(P, k))/m_r(q(P, k)) = g(P; k),$$

where $q(P, k)$ is the cube in $R_r(P)$, parallel to the axes, with center P and length k of the edge, and m_r designates the r-dimensional Lebesgue measure. It is known that $\lim_{k \to 0} g(P; k) = 1$ at almost every point of E. The author now proves that there exists a closed part E^* of E with $m_n(E - E^*)$ arbitrarily small and such that the convergence $g(P; K) \to 1$ is uniform on E^*.

A. Rosenthal.

Pagni, Mauro. Su un teorema relativo all'esistenza di soluzioni per un sistema di n equazioni ad n incognite. Rend. Sem. Mat. Univ. Padova **18**, 234–238 (1949).

The author presents generalizations of the well-known Brouwer fixed point theorem. Let C be the n-dimensional cube $0 \leqq x_i \leqq 1$, $i = 1, \cdots, n$. A function $f(x_1, \cdots, x_n)$, defined on C, is termed lower semi-continuous in the strong sense if it is lower semi-continuous and if for every point $P \in C$ and every pair of positive numbers ϵ and ζ there exists a positive number $r < \zeta$ such that $f(x_1, \cdots, x_n) < f(P) + \epsilon$ on the $(n-1)$-sphere with center P and radius r. The author proves the following theorem. If the functions $f_i(x_1, \cdots, x_n)$, $i = 1, 2, \cdots, n$, are lower semi-continuous in the strong sense in C, and if $f_i < 0$ on the face $x_i = 0$ and $f_i > 0$ on the face $x_i = 1$ of C, then there exists at least one interior point of C where the functions $f_1, \cdots, f_n$ vanish simultaneously. The author proves a similar theorem for functions which possess a property termed strict inferior and linear semi-continuity.

T. Radó (Columbus, Ohio).

Levi, Beppo. Considerations on the Jacobian. Math. Notae **8**, 97–102 (1948). (Spanish)

Let f be a differentiable function, defined on a compact subset D of n-space, and mapping it onto a subset D^* of n-space. Let J be the Jacobian of f. Let E be the set of interior points of D for which: (1) $J(x) = 0$, (2) $f(x)$ is a boundary point of D^*. Observing that J is of Baire class 1, the author proves that E is either void, or is a noncountable set which is dense in itself. If, in addition, the partial derivatives of f are continuous, E is perfect. If these results are applied to the mapping f, obtained by adjoining an appropriately chosen function to a set of $n-1$ real-valued differentiable functions, theorems on partial differential equations follow. For example, if $u(x, y)$ is differentiable in a neighborhood of the origin, then there is a noncountable dense-in-itself set of points for which $yu_x - xu_y = 0$.

R. C. Buck (Madison, Wis.).

Cooper, J. L. B. Convergence of families of completely additive set functions. Quart. J. Math., Oxford Ser. **20**, 8–21 (1949).

The author proves some theorems that may be considered as generalizations of theorems of H. Hahn [Monatsh. Math. Phys. **32**, 3–88 (1922)] and O. Nikodym [ibid. **40**, 427–432 (1933)]. Let $L(X)$ be a completely additive set function with real finite values on a σ-field $\mathbf{K}$ or, more abstractly, on a system $\mathbf{K}$ that is called a "complete Boolean ring" by the author. The total variation of L on X, i.e., the difference between the upper and lower bounds of $L(Y)$ for all $Y \subset X$, is denoted by $|L|(X)$. If $\mathfrak{F}$ is a filter on a set E and $f(x)$ is a function of elements x of E, taking its values in some topological space, then $f(x)$ is said to tend to a point y along the filter $\mathfrak{F}$ if for any neighborhood V of y there is a set $F \in \mathfrak{F}$ such that $f(x) \in V$ if $x \in F$. A filter $\mathfrak{M}$ on $\mathbf{K}$ is called a "mesh" by the author if $\mathfrak{M}$ has a base $\mathfrak{B}$ such that, if $X \in \mathfrak{B}$ and $Y \subset X$, then $Y \in \mathfrak{B}$. Moreover, the author calls a completely additive set function L (defined on $\mathbf{K}$) "absolutely continuous with respect to a mesh $\mathfrak{M}$" if for every $\delta > 0$ there is a set $M \in \mathfrak{M}$ such that $L(X) < \delta$ if $X \subset M$.

Then the author proves the following theorems. (1) Let $L_i(X)$ be a family of completely additive set functions each defined for all elements X of $\mathbf{K}$ and absolutely continuous with respect to a mesh $\mathfrak{M}$ on $\mathbf{K}$; for each X in $\mathbf{K}$ let $L_i(X)$ tend to a limit according to a filter $\mathfrak{F}$ on the set $\mathbf{I}$ of indices $\{i\}$, where the filter $\mathfrak{F}$ has a countable base. Then for all

$\delta>0$ there exist sets $\mathbf{M}(\delta)$ in $\mathfrak{M}$ and $\mathbf{F}(\delta)$ in $\mathfrak{F}$ such that $|L_i|(X)<\delta$ if $X\in\mathbf{M}$ and $i\in\mathbf{F}$. Several special cases of theorem (1) are explicitly stated by the author. Moreover, an extension of theorem (1) is given for the case where the values of the functions $L_i(X)$ are elements in a linear topological space. (2) If a family of completely additive set functions $\{L_i\}$, defined for all elements of $\mathbf{K}$, converges for each X in $\mathbf{K}$ to a limit following a filter $\mathfrak{F}$ in the set of indices $\mathbf{I}$, and if $\mathfrak{F}$ has a countable base, then the limit is a completely additive set function on $\mathbf{K}$. In both the theorems (1) and (2), the condition that $\mathfrak{F}$ has a countable base is essential. (3) If the hypotheses of theorem (1) hold, and if each set of the mesh $\mathfrak{M}$ contains an element Z of $\mathbf{K}$ such that convergence of $L_i(X)$ according to the filter $\mathfrak{F}$ is uniform for all elements X disjoint to Z, then convergence is uniform for all elements of $\mathbf{K}$.

The author applies theorem (1) to the theory of summation of series and obtains the following result that generalizes a theorem of H. Steinhaus [Prace Mat.-Fiz. **22**, 121–134 (1911)]. A matrix which sums all convergent sequences of 0's and 1's to their limits cannot sum all sequences of 0's and 1's. *A. Rosenthal* (Lafayette, Ind.).

Appert, Antoine. Double limite et prolongement. C. R. Acad. Sci. Paris **229**, 865–867 (1949).

There are announced generalizations to more general spaces (in any case more general than topological spaces) of results of Bourbaki and Dieudonné [Revue Sci. **77**, 180–181 (1939)] on double limits, and extensions of continuous functions to limit points of the domain. *R. Arens.*

Mulholland, H. P. Solution of Geöcze's problem for a continuous surface $z=f(x, y)$. Proc. London Math. Soc. (2) **51**, 285–293 (1950).

Mulholland, H. P. On Geöcze's problem for non-parametric surfaces. Trans. Amer. Math. Soc. **68**, 330–336 (1950).

These papers contain the first complete solutions of Geöcze's problems for non-parametric surfaces to appear in print. The two solutions are essentially the same, the second (written two years later) consisting of a simplified arrangement of the first. In the meantime, another solution has been announced by Verčenko [see the following review] and the context shows that it rests again essentially on the same ideas. Given a continuous surface $z=f(x, y)$ defined on a closed polygonal region P of the (x, y)-plane, let $A(f, P)$ denote its Lebesgue area, and let $A^*(f, P)$ denote the lower limit of the areas of polyhedra, inscribed to $z=f(x, y)$, each of which has the form $z=\phi(x, y)$. The theorem proved by Mulholland, and which settles Geöcze's problem for non-parametric surfaces, is that $A(f, P)=A^*(f, P)$. A solution of the corresponding problem for parametric surfaces has been announced by Cesari [Bull. Amer. Math. Soc. **55**, 716–717 (1949)]. As compared with the earlier work of Radó and Huskey, the success of the method used by Mulholland is due entirely to the use of a "local grid": P is divided into small portions in each of which a convenient pair of axes of ξ and η replace those of x and y.
L. C. Young (Madison, Wis.).

Verčenko, I. Ya. Investigations on the theory of the area of surfaces of the form $z=w(x, y)$. Doklady Akad. Nauk SSSR (N.S.) **68**, 5–8 (1949). (Russian)

The author states without proof some properties of a continuous surface $z=w(x, y)$, defined on a rectangle J of the (x, y)-plane, and solves, in particular, the non-parametric case [(I) below] of the problem of Geöcze. [A solution has also been obtained by Mulholland [see the preceding review]; the best previous results were those of Huskey, Duke Math. J. **11**, 333–339 (1944); these Rev. **6**, 45; see also Radó, ibid., 497–506 (1944); these Rev. **6**, 121.] The author's principal results are as follows. (I) There exist inscribed polyhedra $z=w_n(x, y)$ which tend uniformly to $z=w(x, y)$ and whose areas tend to that of $z=(w, y)$. (II) Given a closed subset V of J and $0\leqq\tau<2\pi$, let $s(\xi)$ be the arc length of the part of the section of $z=w(x, y)$ by the plane $x\cos\tau+y\sin\tau=\xi$ determined by the corresponding section of V, and write $S(V, \tau)=\int s(\xi)d\xi$, $S(V)=\max_\tau S(V, \tau)$. Then the area of $z=w(x, y)$ is the limit (finite or infinite) of the sum $\sum S(V_i)$, corresponding to a finite dissection $J=\sum V_i$, where the V_i are closed and without common interior points, provided that the largest diameter of the V_i is made to tend to zero. (III) Let $z=w(x, y)$ be of finite area. Then given $\epsilon>0$, there exists a function θ of (x, y) on J which takes only a finite number of values, such that the area of the surface differs by less than ϵ from the limit as $r\to 0$ of the integrals in (x, y) over J of the expression $\{1+R_\theta\}^{\frac{1}{2}}$ where R denotes, for any angle, the square of the quantity $\tau r^{-1}w(x+r\cos\tau, y+r\sin\tau)-w(x, y)$. There is a similar result, with upper and lower limits in place of limit, when the expression $\{1+R_\theta\}^{\frac{1}{2}}$ is replaced by $\{1+R_\theta+R_{\theta+\tau/2}\}^{\frac{1}{2}}$.
L. C. Young (Madison, Wis.).

Mambriani, Antonio. Sull'approssimazione dell'integrale di Lebesgue per le funzioni di due variabili. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **11(80)** (1947), 201–226 (1949).

With a view to applications to the Geöcze problem in surface area, an approximation theorem is established for a function $f(x, y)$ defined in a rectangle $R\equiv(a\leqq x\leqq b, c\leqq y\leqq d)$. Corresponding to any subdivision σ of $(c\leqq y\leqq d)$ into intervals $l_1, \cdots, l_n$, let η_s denote one of the extremities of l_s and write, for $y\in l_s$ where $s=1, \cdots, n$, $\phi_\sigma(x, y)=f(x, \eta_s)$ and $\psi_\sigma(x, y)=|l_s|^{-1}\int_{l_s}f(x, \eta)d\eta$ (if the latter exists, and say $\psi_\sigma=0$ elsewhere). Then for $p=1$ and for $p=2$ the author proves that if $f\in L^p$ and $\epsilon>0$ then there exists a subdivision σ' of (c, d) into intervals of length less than ϵ such that $\iint_R|f-\phi_{\sigma'}|^p dxdy<\epsilon$, and there exists $\delta>0$ such that $\iint_R|f-\psi_\sigma|^p dxdy<\epsilon$ for every subdivision σ of (c, d) into intervals of length less than δ. *L. C. Young.*

***von Neumann, John. Functional Operators. I. Measures and Integrals.** Annals of Mathematics Studies, no. 21. Princeton University Press, Princeton, N. J., 1950. vii+261 pp. $3.50.

The author's lecture notes on functional operators, despite their limited circulation, were for a long time one of the major sources of measure-theoretic information in the United States. The present volume differs from the first part of the original edition [Institute for Advanced Study, Princeton, 1935] only in that "typographical errors have been corrected and some notations and references have been elaborated." [The first part of the original edition happens not to coincide with this first volume; it contains also the first 35 pages of the second volume.] The first nine chapters contain an extremely thorough and detailed exposition of the classical theory of Lebesgue measure in finite-dimensional Euclidean space. Chapters X and XI, each of which is slightly longer than the first nine chapters together, cover the generalization of the theory to arbitrary measure spaces.

Among the useful material covered is the connection between set functions and point functions, an important and complicated subject which is frequently treated in the trivial one-dimensional case only. The usefulness of the book would be enhanced if it had an index; the somewhat repetitious lecture style of presentation makes it difficult to locate a particular topic. *P. R. Halmos* (Chicago, Ill.).

Tsuji, Masatsugu. On Blichfeld's theorem in the geometry of numbers. Jap. J. Math. **19**, 427–431 (1948).

Let $\mu(e)$ be a nonnegative additive set function defined on Borel sets e in n-dimensional Euclidean space. Suppose that the mean value

$$\lim_{T\to\infty} T^{-n}\int_a^{a+T}\cdots\int_a^{a+T} d\mu(x_1,\cdots,x_n)$$

exists for all $a\geqq 0$ and has a value κ independent of a. If E is a bounded Borel set of positive measure $m(E)$, and if $\epsilon>0$ is given, then by translating E into suitable positions one can obtain sets E' and E'', satisfying

$$\kappa m(E)-\epsilon\leqq\mu(E')\leqq\mu(E'')\leqq\kappa m(E)+\epsilon.$$

When a lattice with density κ is given, and $\mu(e)$ is taken to be the number of lattice points in e, the theorem reduces to a result of Blichfeldt [Trans. Amer. Math. Soc. **15**, 227–235 (1914)]. *C. A. Rogers* (Princeton, N. J.).

Please read Errata p. 871 for correct pagination.

Theory of Series

Petr, K. Sur le calcul effectif des nombres de Bernoulli. Aktuárské Vědy **8**, no. 3, 89–94 (1949).

Writing the Staudt-Clausen theorem in the form

$$(-1)^{r-1}B_r=E_r+\tfrac{1}{6}-\sum_p 1/p,\qquad p-1\,|\,2r,\ p>3,$$

and substituting the recurrence

$$\sum_i((s+1)_{2i+1}+s_{2i+1})B'_{s-i}=0,\qquad B'_k=(-1)^{k-1}B_k,$$

where s_i denotes a binomial coefficient, the author discusses the computation of E_r and B_r. The recurrence is made more usable for numerical work by first computing the numerator corresponding to $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$. Some details are given for the computation of E_{18} and E_{19}. *L. Carlitz.*

Jacobsthal, Ernst. Sur l'inversion d'une série entière. Norske Vid. Selsk. Forh., Trondhjem **20**, no. 17, 62–65 (1948).

Given a function defined by a power series $y=\sum_{n=1}^{\infty}a_nx^n$, $a_1\neq 0$, represent the inverse function by $x=\sum_{n=1}^{\infty}b_ny^n$. The author obtains an explicit formula for b_n in terms of b_{n-1}. He uses classical formulas for the derivative of an inverse function. *T. Fort* (Athens, Ga.).

Jehle, H. Eine Bemerkung zum Konvergenzkriterium von Weierstrass. Math. Z. **52**, 60–61 (1949).

If $a_{n+1}/a_n=(1+c_n)(b_{n+1}/b_n)$, where $\sum|c_n|<\infty$ and if one of $\sum a_n$ and $\sum b_n$ is convergent (or has bounded partial sums) then so is (or has) the other. This fact is used to give a simple proof of a version of the Gauss-Weierstrass test for convergence: if $a_{n+1}/a_n=1+n^{-1}a+c_n$, where $\sum|c_n|<\infty$, then $\sum a_n$ converges if and only if the real part of a is less than -1; in cases of convergence, the convergence is absolute. *R. P. Agnew* (Ithaca, N. Y.).

James, Glenn. A new general method of summing divergent series. Math. Mag. **22**, 235–244 (1949).

Let (a_{ni}) be a triangular matrix of real numbers and let R_n-1 denote the number of changes of type of monotony in the elements $a_{n1}, a_{n2}, \cdots, a_{nn}$ of the nth row. The author calls (a_{ni}) an R-matrix if the following conditions are satisfied: (A) $R_na_{ni}\to 0$ uniformly in i as $n\to\infty$; (B) $\sum_{i=1}^{n}a_{ni}\to 1$ as $n\to\infty$; (C) $\sum_{i=1}^{n}a_{ni}=O(1)$ as $n\to\infty$. With the idea of obtaining a consistent family of regular methods, the author gives a definition of "R-summability" which the reviewer did not find entirely clear, but which, according to correspondence from the author, can be formulated by saying that a series is R-summable to s if it is summable to the same value s by all R-matrices. Several properties of R-matrices are developed. It is shown that every periodic sequence $s_1,\cdots,s_p,s_1,\cdots,s_p,\cdots$ is R-summable to $(s_1+\cdots+s_p)/p$. *J. D. Hill* (East Lansing, Mich.).

Schur, Zvi. On oscillations of infinite sequences. Riveon Lematematika **3**, 39–41, 53 (1949). (Hebrew. English summary)

In connection with work of W. A. Hurwitz [Amer. J. Math. **52**, 611–616 (1930)] and R. P. Agnew [Amer. J. Math. **61**, 683–699 (1939); these Rev. **1**, 10], the author proves the following. A matrix a_{nk} of complex constants transforms each bounded complex sequence x_n into a sequence $y_n=a_{n0}x_0+a_{n1}x_1+\cdots$ for which

$$\limsup_{m,n\to\infty}|y_m-y_n|\leqq\limsup_{m,n\to\infty}|x_m-x_n|$$

if and only if a_{nk} is representable as the sum $a'_{nk}+a''_{nk}$ of two matrices where a'_{nk} is a columnwise-constant conservative matrix and a''_{nk} is a multiplicative matrix for which $\limsup_{n\to\infty}\sum_{k=0}^{\infty}|a''_{nk}|\leqq 1$. *R. P. Agnew* (Ithaca, N. Y.).

Sonnenschein, Jakob. Sur les séries divergentes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 594–601 (1949).

Following a presentation of notations and remarks involving sequences $s_0, s_1, s_2, \cdots$ for which s_n has the form $s_n=a_1z_1^n+a_2z_2^n+\cdots+a_pz_p^n$, the author defines a family of methods for evaluation of sequences. Let $f(z)$ be analytic at the origin, and for each exponent $n=0, 1, 2, \cdots$ let coefficients a_{nk} be defined by $[f(x)]^n=a_{n0}+a_{n1}z+a_{n2}z^2+\cdots$. A sequence s_n is evaluable to L by the method F involving the transformation $\sigma_n=\sum_{k=0}^{\infty}a_{nk}s_k$ if $\sigma_n\to L$ as $n\to\infty$. The particular function $f(z)=(1+z)/2$ leads to the Euler exponential method of order $\frac{1}{2}$; and $f(z)=e^{z-1}$ leads to the matrix method resulting from replacing the continuous parameter by an integer in the Borel exponential method. *R. P. Agnew* (Ithaca, N. Y.).

Obrechkoff, Nikola. Sur la sommation uniforme par les moyennes arithmétiques des séries. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **40**, 139–172 (1944). (Bulgarian. French summary)

Hardy [Proc. London Math. Soc. (2) **4**, 247–265 (1906)] proved that if $\sum a_n$ is summable C_1, then the power series $\sum a_nx^n$ is uniformly summable C_1 over each interval $-1<\lambda\leqq x\leqq 1$. This theorem is generalized in several respects. Let k be a positive integer, and let $\sum a_n$ be summable C_k. Let $\varphi_0(x), \varphi_1(x), \cdots$ be a sequence of functions such that, for each x in a bounded closed set E, $|\varphi_n(x)|<M$ and $\sum_{n=0}^{\infty}n^p|\Delta^{p+1}\varphi_n(x)|<M$; $p=0, 1, \cdots, k$. Then $\sum a_n\varphi_n(x)$ is uniformly summable C_k over E. An application shows that if $\sum a_n$ is summable C_k, then $\sum a_nx^n$ is uniformly summable C_k over each closed subset of a Stolz sector in the unit circle.

Analogous applications are given for Dirichlet series and Lambert series. *R. P. Agnew* (Ithaca, N. Y.).

Teghem, J. Quelques théorèmes abéliens. Bull. Soc. Roy. Sci. Liége **17**, 257–262 (1948).

Let $E(r)$ denote the Euler transformation

$$t_n=\sum_{k=0}^{n}\binom{n}{k}r^k(1-r)^{n-k}s_k,$$

where r may be complex. Agnew [Amer. J. Math. **66**, 313–338 (1944); these Rev. **6**, 46] proved that $E(r)$ is included in the Silverman-Tamarkin [Math. Z. **29**, 161–170 (1928)] generalization of the Abel method, if and only if $\Re r>0$. The author gives a proof of this theorem, along with two generalizations, by methods differing from those of Agnew. *J. D. Hill* (East Lansing, Mich.).

Meyer-König, W. Untersuchungen über einige verwandte Limitierungsverfahren. Math. Z. **52**, 257–304 (1949).

Let $0<\alpha<1$. A series $\sum u_n$ with partial sums s_n is summable T_α to σ if the series in

$$(1)\qquad t_n(\alpha)=(1-\alpha)^{n+1}\sum_{k=n}^{\infty}\binom{k}{n}\alpha^{k-n}s_k,\qquad n=0,1,2,\cdots,$$

converge to numbers $t_n(\alpha)$ such that $t_n(\alpha)\to\sigma$ as $n\to\infty$. Writing (1) in the form

$$(2)\qquad t_n(\alpha)=(1-\alpha)^{n+1}\sum_{k=0}^{\infty}\binom{n+k}{n}\alpha^k s_{n+k}$$

prompts the author to define the methods S_α of summability defined by the similar transformation

$$(3)\qquad s_n(\alpha)=(1-\alpha)^{n+1}\sum_{k=0}^{\infty}\binom{n+k}{n}\alpha^k s_k.$$

Properties of these methods T_α and S_α are obtained, and the methods are related to the Euler-Knopp methods E_p and the Borel exponential method B. The methods T_α are among the "circle methods" defined by Hardy and Littlewood [Rend. Circ. Mat. Palermo **41**, 36–53 (1916)]. It seems that the first systematic study of these methods is in the thesis of R. Wais [Das Taylorsche Summierungsverfahren, Tübingen, 1935]. Parts of this thesis were rediscovered by P. Vermes [Amer. J. Math. **71**, 541–562 (1949); these Rev. **10**, 699]. The present paper reproduces much but not all of the thesis, with proofs and with appropriate references to Wais.

The author, following Wais, determines the region in which the geometric series is summable T_α, and shows that for suitably restricted sequences s_n, the T_α transform of the T_β transform of s_n is the T_γ transform of s_n where $\gamma=\alpha+\beta-\alpha\beta$. Restrictions on s_n are necessary to justify a change of order of summations to obtain the result. It is concluded that if $0<\beta<\gamma<1$ and $T_\beta s_n=s$ (i.e., s_n is summable T_β to s) and s_n possesses a T_γ transform, then $T_\gamma s_n=s_0$. Consistency of T_β and T_γ follows. When s_k is suitably restricted and (1) holds, a particular series of the form $\sum c_{nk}s_k(\alpha)$ is shown to be summable B to s_k. Again following Wais, the author shows that the sequence $0, s_1, s_2, \cdots$ is summable T_α to S whenever $s_1, s_2, \cdots$ is, and that the converse holds when $\frac{1}{2}<\alpha<1$ but not when $0<\alpha\leqq\frac{1}{2}$. In the case of S_α, this proposition and its converse hold for each α in $0<\alpha<1$. If $0<\beta<\gamma<1$, the methods S_β and S_γ are overlapping; neither includes the other. With suitable relations among the parameters, the matrix formulas $T_\alpha S_\beta=S_\gamma$, $E_pT_\alpha=S_\alpha$, $S_\alpha E_p=S_\beta$, $T_\alpha E_p=E_qT_\beta$ are obtained and used. The methods B, E_p $(p>0)$, T_α $(0<\alpha<1)$ and S_β $(0<\beta<1)$ are shown to be consistent. Several theorems involve summability of sequences s_n for which $n^{-1}s_n\to 0$ as $n\to\infty$. If a bounded sequence is summable by one of the methods E_p $(p>0)$, B, T_α $(0<\alpha<1)$, S_α $(0<\alpha<1)$, then it is also summable to the same value by each one of these methods. *R. P. Agnew* (Ithaca, N. Y.).

Meyer-König, W. Die E_p- und S_α-Summierbarkeit einer Potenzreihe an der Konvergenzgrenze. Math. Z. **52**, 344–354 (1949).

With the notation of the preceding review, nine theorems give information about the B, E_p and S_α transforms of sequences s_n belonging to special classes. If $f(z)=\sum s_n z^n$ has a positive radius of convergence and $0<p<1$, then $E_p s_n=s$ if and only if the expansion of $f(z^{-1})$ in powers of $[z+(2^p-1)]^{-1}$ converges to s when $z=1$. There is a similar condition for $S_\alpha s_n=s$, and a condition for $Bs_n=s$ which involves a Fourier sine integral of $t^{-1}f((1+it)^{-1})$. The remaining theorems treat transforms of sequences for which $f(z)$ has stated properties such as $(1-z)^k f(z)\to A$ as $z\to 1$ over a specified neighborhood of $z=1$. *R. P. Agnew.*

Lorentz, G. G. Direct theorems on methods of summability. Canadian J. Math. **1**, 305–319 (1949).

The author considers regular Toeplitz methods of summability A and proposes to study the problem of imposing further "simple" conditions on A sufficient to guarantee that the convergence fields of all such A's will at least contain a certain non-null class of divergent sequences. Questions of this general character have been treated previously by the author [Ber. Math.-Tagung Tübingen, 1946, pp. 97–99 (1947); Acta Math. **80**, 167–190 (1948); these Rev. **9**, 27; **10**, 367], Agnew [Ann. of Math. (2) **46**, 93–101 (1945); Bull. Amer. Math. Soc. **52**, 128–132 (1946); these Rev. **6**, 150; **7**, 292], Hill [Duke Math. J. **9**, 373–381 (1942); Bull. Amer. Math. Soc. **50**, 227–230 (1944); these Rev. **3**, 295; **5**, 236], Wilansky [see the following review] and others. The method adopted in this paper begins with the notion of the characteristic function $\omega(n)$ of a (finite or infinite) sequence $n_1<n_2<\cdots$ of positive integers, defined for all $n\geqq 0$ as the number of n_ν satisfying $n_\nu\leqq n$. The generic notation $\Omega(n)$ is used to denote a nondecreasing and unbounded positive function defined for $n\geqq 0$. For such a function Ω the class $C_1(\Omega)$ consists of all real bounded sequences $\{s_n\}$ for which the set of indices $n_1<n_2<\cdots$ corresponding to nonzero s_n has a characteristic function $\omega(n)\leqq\Omega(n)$. The class $C_2(\Omega)$ is composed of all real $\{s_n\}$ for which $S_n=s_1+\cdots+s_n=O[\Omega(n)]$. Then $\Omega(n)$ is called a summability function of the first kind or of the second kind for the method A if all sequences of $C_1(\Omega)$ or $C_2(\Omega)$, respectively, are summable-A. Among the results are the following. (a) In order that $\Omega(n)$ be a summability function of the first kind for the method $A\sim(a_{mn})$ it is necessary and sufficient that $\lim_m\sum_{\nu=1}^{\infty}|a_{mn_\nu}|=0$ for every sequence $\{n_\nu\}$ whose characteristic function $\omega(n)\leqq\Omega(n)$; every sequence in $C_1(\Omega)$ is then summable-A to zero. (b) A method A possesses summability functions of the first kind if and only if $\max_n|a_{mn}|\to 0$ as $m\to\infty$. [This condition has been shown by Agnew (in the second reference above) to be sufficient in order that A be stronger than convergence.] (c) If $\Omega(n)$ is a summability function of the first kind for a method A, then $u_n=s_n-s_{n-1}=o[1/\Omega(n)]$ is not a Tauberian condition for A. (d) If there exists a matrix (b_{mn}) such that $b_{mn}\geqq 0$, $b_{mn}\downarrow$ for each fixed m, $b_{m0}\to 0$, $|a_{mn}|\leqq b_{mn}$, $\sum_{n=0}^{\infty}b_{mn}\leqq M$ for

all m, then all functions $\Omega(n)=o(n)$ are summability functions of the first kind for A. (e) If A has a summability function of the first kind $\Omega(n)\neq o(n)$, then A is a gap method, that is, to every $\epsilon>0$ and every $C>0$ there exist arbitrarily large values of n' such that $\sum_{n'\leq n\leq Cn'}|a_{mn}|<\epsilon$ for all m. (f) All summability functions of the first kind for the Abel method, the Cesàro methods of positive order, and the Hausdorff methods corresponding to an absolutely continuous generating function are contained in the formula $\Omega(n)=o(n)$; for the methods of Euler and Borel they are contained in $\Omega(n)=o(n^{\frac{1}{2}})$. The methods of Riemann and Lambert are also discussed along with several results bearing on summability functions of the second kind.

J. D. Hill (East Lansing, Mich.).

Wilansky, Albert. A necessary and sufficient condition that a summability method be stronger than convergence. Bull. Amer. Math. Soc. **55**, 914–916 (1949).

A method of summation A corresponding to a linear transformation (1) $y_n=\sum_r a_{nr}x_r$ is called conservative if y_n is convergent whenever x_r is so, and reversible if the system of equations (1) has a unique solution x_r for any convergent sequence y_n, which is then given by a linear transformation with a matrix B. The author shows that (i) a reversible conservative method A sums some unbounded sequences whenever it sums some divergent sequences and deduces from (i) that (ii) this is the case if and only if $\|B\|=+\infty$. [In fact, (i) is true without assumption of reversibility. For regular methods the corresponding theorem has been stated by Mazur and Orlicz [C. R. Acad. Sci. Paris **196**, 32–34 (1933)] and proved by Darevsky [Bull. Acad. Sci. URSS. Sér. Math. [Izvestiya Akad. Nauk SSSR] **10**, 97–104 (1946); these Rev. **7**, 517], and for conservative methods by K. Zeller [Tübingen Dissertation, 1949].]

G. G. Lorentz (Toronto, Ont.).

Wilansky, Albert. An application of Banach linear functionals to summability. Trans. Amer. Math. Soc. **67**, 59–68 (1949).

A summability matrix $A=(a_{nk})$ is called multiplicative m if the A-limit of a convergent sequence is m times its ordinary limit. Matrices with $m\neq 0$ and $m=0$ show quite different behavior. The author extends this dichotomy to all conservative matrices A, calling A co-regular or co-null if $\lim_{n\to\infty}\sum_k a_{nk}-\sum_k \lim_{n\to\infty} a_{nk}\neq 0$ or $=0$, respectively. Using the method of Mazur [Studia Math. **2**, 40–50 (1930)] he proves, mostly for normal conservative matrices, theorems on consistency and gives characteristic properties of matrices of type M (whose columns are complete in l^1). He shows that some properties (co-regularity, type M) are invariant for all normal conservative matrices with the same field of summability. *G. G. Lorentz* (Toronto, Ont.).

Tricomi, Francesco. Sui simboli O ed o e la teoria dei limiti. Univ. e Politecnico Torino. Rend. Sem. Mat. **8**, 161–166 (1949).

In addition to general remarks, the paper contains a proof of the following theorem. Let $a_n-a_{n-1}=\lambda n^{\alpha}+o(n^{\alpha})$, where λ is real and $\alpha>-1$; then $a_n/n=\lambda(\alpha+1)^{-1}n^{\alpha}+o(n^{\alpha})$.

R. P. Boas, Jr. (Providence, R. I.).

Goldoni, Gino. Sulla coincidenza dei concetti di sommabilità ordinaria e secondo Poisson-Abel delle serie numeriche. Atti Sem. Mat. Fis. Univ. Modena **3**, 14–17 (1949).

The following Tauberian theorem for power-series summability is proved. If a series $\sum a_n$ for which $\sum a_n x^n\to L$ as $x\to 1-$ satisfies, for some pair of positive constants s and t for which $s>1$, the two Tauberian conditions $\lim n^{1/s}a_n=0$ and $|\sum_{n=0}^{N}na_n|<N^{1/(s+t)}$, $N\geqq N_0$, then $\sum a_n$ converges to L.

R. P. Agnew (Ithaca, N. Y.).

Edwards, R. E. A Tauberian theorem. J. London Math. Soc. **24**, 223–229 (1949).

Let $1\leqq p<\infty$, $1/p+1/q=1$, and suppose that $\phi(t)$, $Q(t)$ are real and belong to $L_p(-\infty,\infty)$ and $L_q(-\infty,\infty)$, respectively. Suppose also that $\phi(t)\neq 0$ for almost all t and that, for a real sequence $\{x_n\}$, the functions $\phi(x_n-t)Q(t)$ have constant signs (depending on n) for $-\infty<t<\infty$. The main theorem of the paper shows that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\phi(x_n-t)Q(t)dt=0$$

implies

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\psi(x_n-t)Q(t)dt=0$$

for all functions $\psi(t)$ of L_p. *H. R. Pitt* (Belfast).

Forder, H. G. The Euler-Maclaurin formula. Math. Gaz. **33**, 172–176 (1949).

Emde, Fritz. Zur Passmethode. Arch. Elektr. Übertragung **2**, 211–214 (1948).

Ist T ein Sattelpunkt [Pass] der analytischen Funktion $Q(t)$ mit $q=Q''(T)\neq 0$, so ist unter gewissen Bedingungen das Integral $\int e^{Q(t)}dt$, erstreckt über einen Talweg, der über den Pass lauft, approximatif gleich

$$e^{Q(T)}i(2\pi/q)^{\frac{1}{2}}(1-3Bq^{-1}+15Dq^{-2}-\cdots),$$

wo die Zahlen $B=\frac{1}{8}a^2+\frac{1}{24}b$ und $D=\frac{1}{128}a^4+\frac{1}{16}a^2b+\frac{1}{48}ac+\frac{1}{240}d$ definiert sind durch die Entwicklung

$$2q^{-1}\{Q(T+\vartheta)-Q(T)\}=\vartheta^2-a\vartheta^3-b\vartheta^4-c\vartheta^5-d\vartheta^6-\cdots.$$

Anwendungen auf den Kehrwert der Fakultät und auf die Zylinderfunctionen. *J. G. van der Corput* (Amsterdam).

Emde, Fritz. Zur Passmethode bei Pässen mit drei Tälern. Arch. Elektr. Übertragung **2**, 214–217 (1948).

Anschliessend an die vorige Arbeit gibt der Verfasser hier die analogen Formeln unter der Voraussetzung, dass $Q'(T)$ und $Q''(T)$ beide verschwinden, aber $Q'''(T)\neq 0$ ist. Als Beispiel behandelt er die Zylinderfunktionen von gleichem Index und Argument. *J. G. van der Corput* (Amsterdam).

de Kok, F. On the validity of an asymptotic formula. Simon Stevin **26**, 214–217 (1949).

If $0<\mu<1$ and $f(x)$ is of bounded variation in the interval $0<x<1$, then

$$\lim_{\omega\to\infty}\omega^{\mu}\int_0^1 f(x)e^{i\omega x}x^{\mu-1}dx=e^{\frac{1}{2}\pi i\mu}\Gamma(\mu)f(0+)$$

[P. Hartman, Amer. J. Math. **62**, 115–121 (1940); A. Wintner, J. Math. Physics **24**, 127–130 (1945); these Rev. **1**, 140; **7**, 246]. The author shows that the condition of bounded variation may be replaced by the weaker condition that $f(0+)$ exists, $f(x)$ is integrable over $(0,1)$ and

$$\lim_{\sigma\to 0}\sigma^{-\mu}\int_0^{1-\sigma}|f(x+\sigma)-f(x)|x^{\mu-1}dx=0.$$

J. G. van der Corput (Amsterdam).

Kolden, Kjell. Continued fractions and linear substitutions. Arch. Math. Naturvid. **50**, no. 6, 141–196 (1949).

The paper is concerned with the representation of continued fractions as products $M_1 \cdot M_2 \cdot M_3 \cdot \cdots$ of two by two matrices. Numerous formulas are derived, e.g., the Euler-Minding formula. Applications are made to continued fractions of particular types: symmetric continued fractions, periodic continued fractions, continued fractions for equivalent numbers of A. Hurwitz. The theory is extended to products of n by n matrices. *H. S. Wall* (Austin, Tex.).

Lane, R. E., and Wall, H. S. Continued fractions with absolutely convergent even and odd parts. Trans. Amer. Math. Soc. **67**, 368–380 (1949).

This paper is concerned with the continued fraction $f(c) = K_{p=0}^{\infty} c_{p-1}/1$ $(c_0 = 1)$ with approximants $f_0 = 0$, $f_1 = 1$, $f_2 = 1/1 + c_1, \cdots$. If the series $f_0 + \sum (f_p - f_{p-1})$ converges absolutely, $f(c)$ is said to converge absolutely. (i) It is shown that if the even and odd parts of $f(c)$ converge absolutely, $f(c)$ converges if and only if some $c_p = 0$ or the series $\sum |b_p|$ diverges, where $b_1 = 1, c_p = 1/b_p b_{p+1}, p = 1, 2, \cdots$. This result is obtained by a consideration of the linear fractional transformation $w = T_p(z)$ which carries the points $\infty, 0$, and 1 into f_{p-1}, f_p, f_{p+1}, respectively. (ii) A theorem connecting the fixed points of T_p with the convergence and value of $f(c)$ is proved. (iii) A necessary and sufficient condition for the absolute convergence of $f(c)$ in the neighborhood of the origin is given. (iv) If there exist nonnegative numbers $r_1, r_2, r_3, \cdots$, such that $r_1|1+c_1| \geqq |c_1|$, $r_2|1+c_1+c_2| \geqq |c_2|$, $r_p|1+c_{p-1}+c_p| \geqq r_p r_{p-1}|c_{p-1}| + |c_p|$, $p = 3, 4, \cdots$ (where it is agreed that inequality holds in the first two relations if $c_p \neq 0$, $p = 1, 2, \cdots$), then $f(c)$ converges if and only if either some $c_p = 0$ or $c_p \neq 0$ and the series $\sum |b_p|$ diverges. Furthermore, if the above inequalities are satisfied, the even and odd parts of $f(c)$ converge absolutely. (v) Convergence regions for $f(c)$ are obtained from geometrical properties of the above inequalities. (vi) Sufficient conditions for the absolute convergence of $f(c)$ are established.

E. Frank (Chicago, Ill.).

Special Functions

Duff, Georges F. D. Théorème de composition pour l'équation de Truesdell. C. R. Acad. Sci. Paris **229**, 1195–1197 (1949).

The author considers a problem proposed by the reviewer [An Essay Toward a Unified Theory of Special Functions, Princeton University Press, 1948, § 22; these Rev. **9**, 431]: given two functions $F_1(x, \alpha)$ and $F_2(x, \alpha)$ satisfying the F-equation $\partial F(z, \alpha)/\partial z = F(z, \alpha+1)$, to evaluate explicitly double generating expansions of the form $\sum F_1(x, \alpha+n) F_2(y, \alpha+n) z^n/n!$. For the important class of F-functions representable as contour integrals of the type $\int w^{\alpha} e^{xw} f(w) dw$, the author gives four expressions for the sum in question, two being simple contour integrals and two double contour integrals. The equality of three of these integrals constitutes a convolution theorem for this class of F-functions. The proof rests upon the uniqueness theorem of the reviewer [op. cit., § 11]. The author states that his theorem has numerous applications to familiar special functions; it is evident that the formulae involving Bessel and Laguerre functions mentioned in this connection by the reviewer [op. cit., § 22] are included as special cases of the author's result. *C. Truesdell* (Washington, D. C.).

Sales Vallés, Francisco de A. On the first law of errors of Laplace. Collectanea Math. **1**, 85–135 (1948). (Spanish)

Reprint of the author's Barcelona thesis [1947]; cf. these Rev. **9**, 349. *W. Feller* (Ithaca, N. Y.).

Bruwier, L. Sur l'application d'un système de nombres hypercomplexes à l'étude des sinus hyperboliques d'ordre supérieur. Mathesis **58**, 216–222 (1949).

In this paper the author discusses an algebraic approach to the theory of the so-called trigonometric functions of higher order. With a primitive nth root of unity, j, he sets up a system of hypercomplex numbers, or a linear algebra, with units $1, j, j^2, \cdots, j^{n-1}$, and shows that the generalised hyperbolic sine functions are the components, in this algebra, of e^{jt}. Various properties of the functions in question are derived from this relation. *A. Erdélyi.*

Bruwier, L. Sur une généralisation des fonctions hyperboliques et des fonctions circulaires d'ordre supérieur. Bull. Soc. Roy. Sci. Liége **18**, 169–183 (1949).

A fundamental system of solutions $x_i(t)$ of a linear differential equation with constant coefficients has many features which are generalisations of trigonometrical formulas. Thus $x_i'(t)$ can be expressed as a linear combination of the $x_j(t)$, the value of the Wronskian is known (this corresponds to $\sin^2 t + \cos^2 t = 1$), and $x_i(u+t)$ can be expressed as $\sum_{j,k} a_i^{jk} x_j(u) x_k(t)$, thus giving an addition theorem. The author elaborates this remark and obtains many specific formulas by operational calculus; and in particular he applies his ideas to the differential equation $x^{(m)}(t) \pm x(t) = 0$.

A. Erdélyi (Pasadena, Calif.).

Cerrillo, Manuel. Inversion of Laplace transforms of type $F(S, \sqrt{S^2+B^2})e^{-A\sqrt{S^2+B^2}}$. Comisión Impulsora y Coordinadora de la Investigación Científica (Mexico). Anuario **1947**, 31–86 (1949). (Spanish)

The inversion of Laplace transforms of the form $\exp(-A(s^2+B^2)^{\frac{1}{2}})$ times a rational function of s and $(s^2+B^2)^{\frac{1}{2}}$ is accomplished by the introduction of $s+(s^2+B^2)^{\frac{1}{2}}$ as a new variable of integration in the complex inversion formula, when decomposition into partial fractions of the rational function and integrals of Gilbert's type for Lommel's functions of two variables [Watson, A Treatise on the Theory of Bessel Functions, 2d ed., Cambridge University Press, Macmillan, New York, 1944, § 16.58; these Rev. 6, 64] give the answer. Since the expressions so obtained are not suitable for numerical calculation, asymptotic and approximate expressions and graphical methods are developed for certain combinations of Lommel functions. *A. Erdélyi.*

Bouwkamp, C. J. On the evaluation of certain integrals occurring in the theory of the freely vibrating circular disk and related problems. Nederl. Akad. Wetensch., Proc. **52**, 987–994 = Indagationes Math. **11**, 366–373 (1949).

The double integral

$$\int_0^1 P_{2n+1}((1-\rho'^2)^{\frac{1}{2}}) \rho' d\rho' \cdot (2\pi)^{-1} \int_0^{2\pi} (\rho^2 - 2\rho\rho' \cos\theta' + \rho'^2)^{\frac{1}{2}m - \frac{1}{2}} d\theta'$$

occurs in the study of the radiation of sound from a freely vibrating circular disc, and also in the theory of diffraction by circular discs and apertures; m and n are nonnegative integers and $0 \leqq \rho \leqq 1$. The author derives a number of relations for the double integral, and evaluates it in terms of a hypergeometric series with the variable ρ^2. In his work

he uses the connection of his double integral with potential and wave theory. *A. Erdélyi* (Pasadena, Calif.).

Blanuša, Danilo. Some types of integral theorems for Bessel functions. Rad Jugoslav. Akad. Znanosti i Umjetnosti **271**, 83–143 (1948). (Croatian)

Let Λ_n be the entire function occurring in the definition of Bessel functions of the first kind (in the notation used by Jahnke and Emde). After some preliminary matter, the author obtains an expression for the mth derivative of $\Lambda_n(c(t^2-x^2)^{\frac{1}{2}})$ with respect to x. This he uses to integrate by parts in

$$I_{mn}=\int_x^t x^m\Lambda_n(c(t^2-x^2)^{\frac{1}{2}})dx.$$

By repeated steps he succeeds either in evaluating this integral in a finite number of steps or, at worst, reducing it to I_{00} and I_{01}. He also computes derivatives of I_{m0} with respect to t. *A. Erdélyi* (Pasadena, Calif.).

Blanuša, Danilo. Une espèce de théorèmes intégraux des fonctions de Bessel. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **2**, 39–61 (1949).

A condensed translation of the paper reviewed above. *A. Erdélyi* (Pasadena, Calif.).

Abbott, Wilton R. Evaluation of an integral of a Bessel function. J. Math. Physics **28**, 192–194 (1949).

The author considers the integral

$$v(t,n)=2n\int_0^t\{J_{2n}(\gamma t)\}t^{-1}dt,$$

which is encountered in some technical problems [cf. Gardner and Barnes, Transients in Linear Systems, v. 1, Wiley, New York, 1942, pp. 310–317; these Rev. **4**, 150]. This integral may be expanded in a finite series of Bessel functions, for instance by means of successive integrations by parts. A convenient medium for carrying out the expansion is given by the Lebesgue form of the Laplace transformation. The result is $v(t,n)=1-2(\gamma t)^{-1}\sum_{k=1}^{n}(2k-1)J_{2k-1}(\gamma t)$. Another expansion may be found by means of the Λ functions, defined by $\Lambda_p(\gamma t)=2^p p!J_p(\gamma t)/(\gamma t)^p$. They are tabulated in Jahnke and Emde's Tables through the eighth order, and in J. Math. Physics **23**, 45–60 (1944) [these Rev. **5**, 245] through the twentieth order. The author finds

$$v(t,n)=1-\sum_{p=0}^{n-1}\frac{(-1)^p(2n-p-1)!}{p!(n-p-1)!(n-p)!}\Lambda_{n-p}(\gamma t).$$

He gives the expansions for values of n up to 9.

S. C. van Veen (Delft).

Mursi, Z. On the relation of the Airy and allied integrals to the Bessel function. Proc. Math. Phys. Soc. Egypt **3**, no. 4, 23–38 (1948).

The Airy and allied integrals are defined by

$$\text{Ai } x=\pi^{-1}\int_0^\infty\cos(\tfrac{1}{3}s^3+xs)ds,$$

$$\text{Bi } x=\pi^{-1}\int_0^\infty\{\exp(-\tfrac{1}{3}s^3+xs)+\sin(\tfrac{1}{3}s^3+xs)\}ds.$$

They are two independent solutions of the differential equation $y''-xy=0$. Nicholson [Philos. Mag. (6) **18**, 6–17 (1909)] has found that these integrals are related to the Bessel functions of order $\pm\frac{1}{3}$, $\pm\frac{2}{3}$ by

$$J_{\pm\frac{1}{3}}(\tfrac{2}{3}x^{\frac{3}{2}})=\tfrac{1}{3}x^{-\frac{1}{2}}\{3\text{ Ai }(-x)\mp3^{\frac{1}{2}}\text{ Bi }(-x)\},$$
$$J_{\pm\frac{2}{3}}(\tfrac{2}{3}x^{\frac{3}{2}})=\tfrac{1}{3}x^{-\frac{1}{2}}\{3^{\frac{1}{2}}\text{ Bi}'(-x)\pm3\text{ Ai}'(-x)\}.$$

By means of Sonine's relation

$$J_m(x)=\frac{2x^{m-n}}{2^{m-n}\Gamma(m-n)}\int_0^1 J_n(xv)v^{n+1}(1-v^2)^{m-n-1}dv,\qquad m>n>-1$$

(in the paper read v^{n+1} in place of v^{m+1} under the integral), $J_m(x)$ is expressed as an integral involving the Airy and allied functions for $m>-\frac{2}{3}$. By means of the recurrence relations for $J_m(x)$ it is evident that for $m=l+\frac{1}{3}$ and $m=l+\frac{2}{3}$ (l being a positive integer or zero) there exist relations of the form

$$\lambda^{\frac{1}{3}m}J_m(\tfrac{2}{3}\lambda^{\frac{1}{2}})=Q_1\cdot(2\lambda\cdot J_{-\frac{1}{3}}(\tfrac{2}{3}\lambda^{\frac{1}{2}}))+Q_2\cdot(2\lambda\cdot J_{-\frac{2}{3}}(\tfrac{2}{3}\lambda^{\frac{1}{2}})),$$

where Q_1 and Q_2 are polynomials in λ. These polynomials may be expressed by 4 standard forms, of which the coefficients $\alpha_{2m,v}$, $\beta_{2m,v}$, $\gamma_{2m-1,v}$, $\delta_{2m-1,v}$ are given in a table for m: 1(1)15; v: 0(1)5. *S. C. van Veen* (Delft).

Darwin, C. G. On Weber's function. Quart. J. Mech. Appl. Math. **2**, 311–320 (1949).

The differential equation (*) $d^2u/dx^2+(x^2/4-a)u=0$ (a and x real) occurs in a number of wave problems. Its solutions can be expressed in terms of the parabolic cylinder function $D_\nu(z)$, where z is a real multiple of $\exp(\pm\pi i/4)$, and $\nu+\frac{1}{2}$ is imaginary. When the tabulation of these functions was taken in hand, J. C. P. Miller and C. W. Jones investigated the various solutions of this equation in order to find the most suitable ones for tabulation, and the reviewer pointed out that Miller's choice can be confirmed on theoretical grounds, in particular from appropriate generalisations of Fourier's integral theorem, when an arbitrary function is expanded in terms of solutions of (*). Much of this work leans more or less heavily on the theory of confluent hypergeometric functions.

In the present paper, the author sets himself the task to develop the subject ab initio, without making use of the theory of confluent hypergeometric functions. The even and odd solutions of (*) are not suitable for tabulation, since for large x they are nearly proportional to one another. The solutions chosen are such that $u_{II}(x)=u_I(-x)$. For positive a, $u_I(x)$ resembles an exponential function near the origin, rises to a large value near $x=2a^{\frac{1}{2}}$, and then oscillates with slowly decreasing amplitude. For large x, the oscillations of u_I and u_{II} are nearly 90° out of phase. For negative a, there is a similar pair of solutions, but they oscillate everywhere. The author gives convergent power series of the solutions, asymptotic series for large x, and yet another set of series for large a. *A. Erdélyi* (Pasadena, Calif.).

Abramowitz, Milton. Asymptotic expansions of spheroidal wave functions. J. Math. Physics **28**, 195–199 (1949).

The author considers, for the cases when m is large with respect to c and when m and c are both large, the characteristic value b of the spheroidal wave equation

$$(1-z^2)y''-2(m+1)zy'+(b-c^2z^2)y=0$$

determined from conditions of finiteness and periodicity over the surface of a spheroid. By the substitution $\mu=(2m+2)^{-1}$; $x=\mu^{-\frac{1}{2}}z^{\frac{1}{2}}$; $\lambda=b\mu$ the equation becomes

$$(1-\mu x^2)y''-xy'+(\lambda-\mu^2c^2x^2)y=0,$$

where μc is small and μ very small. By substituting $y=\sum y_k\mu^k$ and $\lambda=\sum\lambda_k\mu^k$ and by setting the various powers of μ equal to zero the author gets a differential equation of the second order for y_k, which contains $y_0, y_1, \cdots, y_{k-1}$. He obtains $y_0=H_l(x)$, where $l=\lambda_0$. By the substitution $y_1=\sum a_kH_k(x)$ the left-hand side of the equation satisfied by y_1 becomes a finite sum of Hermite polynomials, in which the coefficient of $H_l(x)$ is equal to $\lambda_1-l(l-1)$. Putting this coefficient equal to zero he finds $\lambda_1=l(l-1)$. Similarly he deduces values for $\lambda_2, \lambda_3, \cdots$ from the equations satisfied by $y_2, y_3, \cdots$. The result is

$$b=2(m+1)l+l(l-1)+\frac{(2l+1)c^2}{2(m+1)}-\frac{c^2(6l^2+2l+1)}{2(m+1)^3}+\cdots.$$

For the case that m and c are large and $c/(2m+2)$ is finite he obtains in a similar manner

$$b=(2l+1)\{c^2+(m+1)^2\}^{\frac{1}{2}}-(m+1)+\tfrac{1}{2}(1-2l-l^2)-\frac{(2l+1)(m+1)}{\{c^2+(m+1)^2\}^{\frac{1}{2}}}+\frac{(6l^2+6l+3)(m+1)^2}{4c^2+4(m+1)^2}+\cdots.$$

He gives some numerical examples from which the accuracy of his formulae may be judged. *J. G. van der Corput.*

Koschmieder, Lothar. Verallgemeinerte Ableitungen und hypergeometrische Funktionen. Monatsh. Math. **53**, 169–183 (1949).

This paper has much in common with an earlier memoir by the same author [Acta Math. **79**, 241–254 (1947); these Rev. **9**, 351]. In the first five sections of this paper the application of fractional differentiation to hypergeometric functions both in one and n variables is illustrated by three examples. In the remaining sections the author shows that Euler's transformation of the hypergeometric series can be obtained from the extension of Leibniz's rule for the differentiation of a product to derivatives of fractional order. A similar derivation is given of Kummer's transformation of confluent hypergeometric series and of the linear transformation of some hypergeometric series of two or more variables. *A. Erdélyi* (Pasadena, Calif.).

Jackson, M. On some formulae in partition theory, and bilateral basic hypergeometric series. J. London Math. Soc. **24**, 233–237 (1949).

The author shows that a general formula of D. B. Sears for well-poised basic hypergeometric series can be specialized to yield a relation between a well-poised bilateral basic hypergeometric series and two well-poised basic series of the ordinary type. This in turn can be specialized to the following unpublished result of A. O. L. Atkin and H. P. F. Swinnerton-Dyer, as well as generalizations of it:

$$\sum_{n=-\infty}^{\infty}(-1)^nq^{\frac{1}{2}n(n+1)}\left[\frac{\zeta^{-2n}}{1-z\zeta^{-1}q^n}+\frac{\zeta^{2n+2}}{1-z\zeta q^n}\right]=\frac{\zeta P(\zeta^2,q)}{P(\zeta,q)}\sum_{n=-\infty}^{\infty}\left\{\frac{(-1)^nq^{\frac{1}{2}n(n+1)}}{1-zq^n}\right\}+\frac{P(\zeta,q)P(\zeta^2,q)\prod_{r=1}^{\infty}(1-q^r)^2}{P(z\zeta^{-1},q)P(z,q)P(z\zeta,q)},$$

where $P(z,q)=\prod_{r=0}^{\infty}(1-zq^r)(1-z^{-1}q^{r+1})$. Results of this type can be useful in partition theory. *N. J. Fine.*

Jackson, M. A generalization of the theorems of Watson and Whipple on the sum of the series ${}_2F_2$. J. London Math. Soc. **24**, 238–240 (1949).

There is obtained a general expression for a bilateral series ${}_2H_2$ as the sum of two ${}_2F_2$. By specialization this is reduced to identities of the type:

$${}_2H_2\left[\begin{matrix}a,b,c\\e,f,g\end{matrix};\right]=\frac{2^{b+c-2a}\Gamma(e)\Gamma(f)\Gamma(g)\Gamma(1-a)}{\pi\Gamma(\frac{1}{2}+\frac{1}{2}e-\frac{1}{2}b)\Gamma(\frac{1}{2}+\frac{1}{2}e-\frac{1}{2}c)}\times\frac{\Gamma(1-b)\Gamma(1-c)\Gamma(\frac{1}{2}+a-\frac{1}{2}b-\frac{1}{2}c)}{\Gamma(1+a-\frac{1}{2}e-\frac{1}{2}b)\Gamma(1+a-\frac{1}{2}e-\frac{1}{2}c)}\times[\sin a\pi\cos\tfrac{1}{2}(b-c)\pi+\sin(e-a)\pi\cos\tfrac{1}{2}(b+c)\pi].$$

N. A. Hall (Minneapolis, Minn.).

MacDonald, A. D. Properties of the confluent hypergeometric function. J. Math. Physics **28**, 183–191 (1949).

The confluent hypergeometric function $M(\alpha;\gamma;z)$ is defined as the solution, bounded at the origin, of the second order linear homogeneous differential equation

$$(1)\qquad zd^2M(dz^2+(\gamma-z)dM)dz-\alpha M=0;$$

γ, α, and z are unrestricted. Then

$$M(\alpha;\gamma;z)=\sum_{n=0}^{\infty}\frac{\Gamma(\gamma)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(\gamma+n)}\frac{z^n}{n!}.$$

Equation (1) has a regular singularity at the origin and an irregular singularity at infinity. If γ is not an integer, a second solution is given by $W(\alpha;\gamma;z)=z^{1-\gamma}M(\alpha-\gamma+1;2-\gamma;z)$. If γ is an integer, a second solution is given by

$$W(\alpha;\gamma;z)=M(\alpha;\gamma;z)[\log z+\psi(1-\alpha)-\psi(\gamma)+C]+\sum_{n=1}^{\infty}\frac{\Gamma(n+\alpha)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(n+\gamma)}\frac{B_nz^n}{n!}+(-1)^{\gamma}\sum_{n=0}^{\gamma-2}\frac{\Gamma(\gamma)\Gamma(n+\alpha-\gamma+1)\Gamma(\gamma-n-1)(-1)^n}{\Gamma(\alpha)n!z^{\gamma-n-1}};$$

$$\psi(\alpha)=\Gamma'(\alpha)/\Gamma(\alpha);\quad B_n=\sum_{k=0}^{n-1}\left(\frac{1}{\alpha+k}-\frac{1}{\gamma+k}-\frac{1}{1+k}\right);$$

C is Euler's constant. The more general equation

$$\frac{d^2y}{dx^2}-\frac{a^2x^{2k}+bx^k+c}{(2/k)^2x^2}y=0$$

may be reduced to (1). The solutions are

$$y=x^{\frac{1}{2}(1\pm(a^2+1)^{\frac{1}{2}})}e^{-ax^k/2}\times M\left\{\frac{b}{4a}+\tfrac{1}{2}(1\pm(c+k^2)^{\frac{1}{2}});\ 1\pm(c+k^2)^{\frac{1}{2}});\ ax^k\right\}.$$

Several indefinite integrals, e.g., $\int z^ke^{-z}M(\alpha;\gamma;z)dz$ $(k=0,1)$ may be expressed by means of $M(\alpha;\gamma;z)$ and $(d/dz)M(\alpha;\gamma;z)$. Six-figure tables of $M(\alpha;\gamma;z)$ are given for $\gamma=0.5$; 1.0; 1.5 and 2.0; $\alpha=0.001$, 0.01, 0.05, 0.1, 0.2, 0.25, 0.3, 0.4, 0.5, 0.6, 0.7, 0.75, 0.8, 0.9 and 1.0. Tables of the second solutions for $\gamma=1$, 2, and 3 at the same values of α are also included: $z=0.5(0.5)8.0$. For $z>8$ the asymptotic expansions suffice.

S. C. van Veen (Delft).

Differential Equations

Petrovitch, Michel. Série taylorienne exprimant l'intégrale générale d'une équation différentielle du premier ordre. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **5**, 21–23 (1939).

The author finds an expression derived from the two coefficients A_n and A_{n+1} of a Taylor's series. This expression must be independent of n to make the series a solution of an equation of the form $y'=f(x, y)$. *P. Franklin.*

Petrovitch, Michel. Propriété commune à une multitude d'équations différentielles. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **5**, 49–56 (1939).

The property is that from some particular function of two solutions, and the differential equation, these solutions can be individually found by quadratures. Several examples are given. *P. Franklin* (Cambridge, Mass.).

Mitrinovitch, Dragoslav S. Équations différentielles d'Abel d'ordre supérieur. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **5**, 25–31 (1939).

The author studies special reducible cases of the differential equation $fy^{(n)}=F$, where F is an expression of the second degree and f is one of the first degree in y, y', and the higher derivatives up to $y^{(n-1)}$. *P. Franklin.*

Mitrinovitch, Dragoslav S. Sur une classe d'équations différentielles du premier ordre que l'on rencontre dans divers problèmes de géométrie. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **6**, 99–120 (1939).

For a first order differential equation with terms of the first and second degree in y and y', with coefficients functions of x, a number of cases of integrability are given. *P. Franklin* (Cambridge, Mass.).

Mitrinovitch, Dragoslav S. Quelques propositions relatives à l'équation différentielle de Riccati. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. **6**, 121–156 (1939).

From an integrable equation of the form $y'+y^2=\phi(x)$, new integrable equations of this type are derived, with the new right member having the generality of an arbitrary function of x. *P. Franklin* (Cambridge, Mass.).

Tschakaloff, Ljubomir. Über die Riccatischen Differentialgleichung. Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. **37**, 149–195 (1941). (Bulgarian. German summary)

Expository article. Cf. the author's papers [Rend. Accad. Sci. Fis. Mat. Napoli (3) **31**, 120–122 (1925); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **2**, 13–16 (1925); Giorn. Mat. Battaglini **63**, 139–179 (1925)].

Karapandjitch, G. Deux cas d'intégrabilité de l'équation de la ballistique extérieure. Godišnjak Tehn. Fak. Univ. Beograd. **1946–47**, 5–7 (1949). (Serbian. French summary)

Il s'agit de l'équation de d'Alembert de la forme

$$dy/dx=(1-y^2)/(y+\rho(x)),$$

et l'auteur cite quatre cas d'intégrabilité donnés par d'Alembert. On voit que pour la fonction $\rho(x)=Ae^{nx}+Be^{-nx}+R$ les coefficients doivent remplir la condition nécessaire $R^2=A^2+4AB+4$, où A, B, R sont des constantes. Cependant l'auteur montre que, dans le cas $\rho=Ae^{2x}+Be^{-2x}$, la condition de d'Alembert n'étant pas remplie, l'équation est intégrable. On obtient ce résultat grâce à la transformation de contact $y'=f'x_1$, $y=(f/f')y'-y_1$, $y_1'=f$, en introduisant les conditions nécessaires $ff'+f^2=-f^2$ ou $f'+2f=0$ et $\rho f'=\beta(f^2+1)$. En outre on donne une autre condition, qu'on obtient par la substitution $y+\rho=\alpha^{-1}ze^{-x}$ (α=constante), qui transforme l'équation de la ballistique en $zz'-\alpha e^x(2\rho+\rho')(z+1)=0$, sous la condition

$$\rho'+2\rho+\alpha e^x(\rho^2-1)=0.$$

Enfin on obtient la condition nouvelle pour l'équation de d'Alembert,

$$\rho=-\alpha^{-1}e^{-x}+[1+c\exp(2\alpha e^x)]/[1-c\exp(2\alpha e^x)].$$

Author's summary.

Hukuhara, Masuo. On the expansion of the solution of differential equations in the neighborhood of their singular point. Mem. Fac. Sci. Kyūsyū Univ. A. **4**, 1–7 (1949). (Esperanto)

The problem is that of finding formal solutions of the equations $xy'=y^{m+1}z^nf(x, y, z)$, $xz'=y^mz^{n+1}g(x, y, z)$, where $f(x, y, z)$, $g(x, y, z)$ are regular in the neighborhood of $x=y=z=0$ and $f(0, 0, 0)$, $g(0, 0, 0)$ are both nonzero, having the form $y=u(1+\sum'p_{ijk}x^iu^jv^k)$, $z=v(1+\sum'q_{ijk}x^iu^jv^k)$. Several different cases occur depending upon whether a/b is rational or irrational, positive or negative rational. *R. Bellman* (Stanford University, Calif.).

Hukuhara, Masuo. On singular points of the ordinary differential equation of first order. Mem. Fac. Sci. Kyūsyū Univ. A. **4**, 9–21 (1949). (Esperanto)

Given an equation of the form $x^{a+1}y'=f(x, y)y$, a being a positive integer, $f(x, y)$ regular in the neighborhood of $(0, 0)$, $f(0, 0)\neq 0$, and the formal series satisfying the equation, an important problem is that of establishing the region of convergence of the series. This the author accomplishes under certain assumptions too detailed to relate here, by a combination of analysis and the method of the Tychonoff fixed-point theorem. *R. Bellman.*

Péyovitch, T. Sur les solutions asymptotiques de certaines équations différentielles. Glas Srpske Akad. Nauka **191**, 189–196 (1948). (Serbian. French summary)

The author states and "proves" that the system

$$dx/dt=-l+a_{11}e^x+a_{12}e^y,\quad dy/dt=m+a_{21}e^x+a_{22}e^y,$$

where $l>0$ and $m\geqq 0$ and $a_{ij}(t)$ are continuous in t has a solution $x(t)$, $y(t)$ such that $x\to-\infty$, $y\to-\infty$, $dx/dt\to-l$, $dy/dt\to m$ as $t\to\infty$. [The reviewer notes that $y\to-\infty$, $dy/dt\to m>0$ is impossible.] *N. Levinson.*

Péyovitch, T. Sur une propriété asymptotique de certaines équations différentielles. Glas Srpske Akad. Nauka **191**, 197–199 (1948). (Serbian. French summary)

If the $a_{ij}(t)$ are continuous and bounded for $t>0$, if λ is sufficiently negative and if the system

$$dx_i/dt=\sum_{j=1}^{n}(a_{ij}(t)-\lambda)e^{x_j},$$

$i=1, \cdots, n$, admits a solution for $t>0$, then $x_i(t)\to-\infty$ and $dx_i/dt\to 0$ as $t\to\infty$. *N. Levinson* (Cambridge, Mass.).

El'šin, M. I. The phase method and the classical method of comparison. Doklady Akad. Nauk SSSR (N.S.) **68**, 813–816 (1949). (Russian)

Generalizing the transformation which removes the coefficient of x' in (1): $x''+p(t)x'+q(t)x=0$, the author obtains

the "characteristic operator"

$$J(\theta, p, q) = (\theta - p/2)' + \theta^2 + q - p^2/4$$

previously considered [same vol., 221–224 (1949); these Rev. 11, 110]. This operator is then used to deduce results concerning the oscillation of the solutions of (1).

R. Bellman (Stanford University, Calif.).

El'šin, M. I. On the decremental estimate of amplitudes. Doklady Akad. Nauk SSSR (N.S.) **69**, 7–10 (1949). (Russian)

The author has previously shown that under certain conditions the general solution of $u''+p(t)u'+q(t)u=0$ can be put in the form $u=c_1 \exp(-\frac{1}{2}\int_{t_0}^{t} p\,ds)\cos[\int_{t_0}^{t} w\,ds+c_2]/|w|^{\frac{1}{2}}$, where w is determined by another functional equation of differential type. He now investigates the boundedness or unboundedness of the amplitude, $\exp(-\frac{1}{2}\int_{t_0}^{t} p\,ds)/|w|^{\frac{1}{2}}$.

R. Bellman (Stanford University, Calif.).

Leighton, Walter. On self-adjoint differential equations of second order. Proc. Nat. Acad. Sci. U. S. A. **35**, 656–657 (1949).

This is an abstract of a longer paper to appear elsewhere. It announces theorems concerning the oscillatory behavior of solutions. Some of these theorems are related to results of Wintner [Amer. J. Math. **69**, 251–272 (1947), in particular, p. 263; these Rev. **9**, 35] and of Hille [Trans. Amer. Math. Soc. **64**, 234–252 (1948), in particular, p. 241; these Rev. **10**, 376]. *P. Hartman* (Baltimore, Md.).

Leighton, Walter. The detection of the oscillation of solutions of a second order linear differential equation. Duke Math. J. **17**, 57–61 (1950).

The author first proves that, if

$$\int^{\infty} r^{-1}(x)\,dx = \infty, \quad \int^{\infty} p(x)\,dx = \infty,$$

then the differential equation $(r(x)y')'+p(x)y=0$ is oscillatory on a half-line $[a, \infty)$. If the independent variable x is replaced by $s=\int^x [r(x)]^{-1}dx$, it is seen that this result is contained in that of Wintner [Quart. Appl. Math. **7**, 115–117 (1949); these Rev. **10**, 456]. The same change of variable shows that the second theorem of the paper is a consequence of a result of A. Kneser [Math. Ann. **42**, 409–435 (1893), in particular, p. 415]. The author's logarithmic scale of tests is closely related to the tests of Hille [Trans. Amer. Math. Soc. **64**, 234–252 (1948), in particular, p. 250; these Rev. **10**, 376] and of the reviewer [Amer. J. Math. **70**, 764–779 (1948), in particular, p. 778; these Rev. **10**, 376]. *P. Hartman* (Baltimore, Md.).

Please read Errata p. 811

Levinson, Norman. Criteria for the limit-point case for second order linear differential operators. Časopis Pěst. Mat. Fys. **74**, 17–20 (1949). (English. Czech summary)

Let $p(x)>0$, $q(x)$ be (real) continuous functions on $0 \leqq x < \infty$. The author shows, by a simple method, that the differential equation $(py')'+qy=0$ is of the limit-point type if there exists a positive nondecreasing function $M(x)$ such that $q(x)<M(x)$, $\int^{\infty} p^{-1}M^{-\frac{1}{2}}dx<\infty$ and $|p^{\frac{1}{2}}M'|<cM^{\frac{1}{2}}$, c=constant. [In the special case $p\equiv 1$, this theorem overlaps with results of Hartman and Wintner [Amer. J. Math. **71**, 206–213 (1949); these Rev. **10**, 455], which appeared simultaneously.] *P. Hartman* (Baltimore, Md.).

Levinson, Norman. The inverse Sturm-Liouville problem. Mat. Tidsskr. B. **1949**, 25–30 (1949).

In theorem I it is proved that there can be at most one $P(x)\varepsilon L(0, \pi)$ such that $y''+(\lambda-P(x))y=0$ has two assigned sets of characteristic values for the two pairs of boundary conditions

(1) $y(0)\cos\alpha+y'(0)\sin\alpha=y(\pi)\cos\beta+y'(\pi)\sin\beta=0$,
(2) $y(0)\cos\alpha+y'(0)\sin\alpha=y(\pi)\cos\gamma+y'(\pi)\sin\gamma=0$,

provided that $\sin(\beta-\gamma)\neq 0$. The original result of G. Borg [Acta Math. **78**, 1–96 (1946); these Rev. **7**, 382] required in addition that $\sin\beta\sin\gamma=0$, but on the other hand in some cases did not require the lowest characteristic value to be known. In theorem II are imposed the additional restrictions that $P(\pi-x)=P(x)$ almost everywhere, and $\alpha+\beta=\pi$, and here $P(x)$ is fixed if only the characteristic values for (1) are known; Borg's results related to the special cases $\alpha=0, \frac{1}{2}\pi$. The proof is based on the Cauchy integral method.

F. V. Atkinson (Ibadan).

Čudov, L. A. The inverse Sturm-Liouville problem. Mat. Sbornik N.S. **25(67)**, 451–456 (1949). (Russian)

The author proves by a different method the result of theorem I of the preceding paper of Levinson, reducing it to the more special case considered by Borg [loc. cit. supra].

F. V. Atkinson (Ibadan).

Bargmann, V. On the connection between phase shifts and scattering potential. Rev. Modern Physics **21**, 488–493 (1949).

The differential equation (*) $\phi''+k^2\phi=V(r)\phi$ is considered where $\phi'=d\phi/dr$ and k is a constant parameter. With the potential $V(r)$ small in some sense for large r the solution of (*) vanishing at $r=0$ tends as $r\to\infty$ to a sinusoid with phase shift $\eta(k)$. The author shows by examples that there may be infinitely many $V(r)$, extremely well behaved analytically, with the same phase shift $\eta(k)$, $0<k<\infty$. Also the phase shift does not determine bound states (discrete eigen-values). The author refers to results of the reviewer [Danske Vid. Selsk. Mat.-Fys. Medd. **25**, no. 9 (1949); these Rev. **10**, 710] where criteria are given for the phase shift to determine the potential (for the cases of angular momentum $l=1, 2, \cdots$ as well as $l=0$). *N. Levinson.*

Bargmann, V. Remarks on the determination of a central field of force from the elastic scattering phase shifts. Physical Review. (2) **75**, 301–303 (1949).

[Cf. the preceding review.] Fröberg [same Rev. (2) **72**, 519–520 (1947); Ark. Mat. Astr. Fys. **34A**, no. 28 (1948); these Rev. **10**, 120; and the paper reviewed below] and Hylleraas [Physical Rev. (2) **74**, 48–51 (1948); these Rev. **10**, 538] have developed methods for determining a two-body potential corresponding to a definite l (angular momentum) when the asymptotic phase η of the wave function is known for all wave numbers k. The present author shows that the two potentials $V_1(r)=-6\lambda^2 e^{-\lambda r}/(1+e^{-\lambda r})^2$ and $V_2(r)=-24\lambda^2 e^{-2\lambda r}/(1+3e^{-2\lambda r})^2$ (misprint corrected) give the same phase shifts for $l=0$. Thus $V(r)$ cannot in general be uniquely determined by $\eta(k)$; the opposite statement made by Hylleraas [loc. cit.] may, however, be generally valid if the potentials in question do not give rise to bound states ($V_1(r)$ and $V_2(r)$ above lead to one and the same stationary state). *L. Hulthén* (Stockholm).

Fröberg, Carl-Erik. Calculation of the potential from the asymptotic phase. II. Ark. Mat. Astr. Fys. **36A**, no. 11, 55 pp. (1949).

[For part I cf. the same Ark. **34A**, no. 28 (1948); these Rev. **10**, 120.] The equation $y''+k^2y-[l(l+1)/r^2]y=-V(r)y$, where $y'=dy/dr$, is considered with $V(r)$ small for large r. A solution $y=0$ for $r=0$ has then the asymptotic behaviour as $r\to\infty$ given by $\sin(kr-\frac{1}{2}l\pi+\delta)$ where δ depends on k for any fixed integer l. The problem is to determine $V(r)$ when $\delta(k)$ is known for $k\geqq 0$. Using the "variation of constants" formula with $-V(r)y$ treated as a right member a successive approximations procedure is formulated, and a number of results obtained and several cases of physical interest worked out. The author points out that convergence is not demonstrated. [Results of V. Bargmann [see the two preceding reviews] show that without further hypotheses the phase $\delta(k)$ does not determine $V(r)$ uniquely. Sufficient conditions for $\delta(k)$ to determine $V(r)$ have been given by the reviewer [reference in the second preceding review].] *N. Levinson* (Cambridge, Mass.).

***Starr, David Wright. The Schrödinger Wave Equation from the Point of View of Singular Integral Equations.** Abstract of a Thesis, University of Illinois, 1940. ii+9 pp.

The author supplies details omitted by Carleman [Ark. Mat. Astr. Fys. **24B**, no. 11 (1934)].

Cartwright, M. L. On nonlinear differential equations of the second order. III. The equation

$$\ddot{x}-k(1-x^2)\dot{x}+x=p\cdot k\lambda\cos(\lambda t+\alpha),$$

k small and λ near 1. Proc. Cambridge Philos. Soc. **45**, 495–501 (1949).

By a combination of analysis and fixed point procedures a rigorous proof is given of the fact that the singular point in the Poincaré plane of the pair of first order differential equations obtained under the assumption of "slowly varying" functions [Appleton and van der Pol] gives an approximation to an actual periodic solution. Moreover, the periodic solution has the same type of stability as the corresponding singular point in the Poincaré plane. [The slowly varying functions are b_1 and b_2, where a solution of the form $b_1\sin\lambda t+b_2\cos\lambda t$ is assumed.] *N. Levinson.*

Nijenhuis, W. A note on a generalized Van der Pol equation. Philips Research Rep. **4**, 401–406 (1949).

The equation $\ddot{v}-\epsilon(1-2\beta v-v^2)\dot{v}+v=0$ is studied by the method of isoclines. For the periodic solution $v=v(t)$, ϵ and β large, it is shown that $-\max v(t)/\min v(t)$ cannot exceed $\frac{1}{3}$. *J. G. Wendel* (New Haven, Conn.).

Butenin, N. V. On the theory of forced oscillations in a nonlinear mechanical system with two degrees of freedom. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 337–348 (1949). (Russian)

The author discusses a mechanical system with two degrees of freedom close to a linear conservative system and undergoing a forced harmonic oscillation whose period is far from the two fundamental periods of the linear conservative system. The basic equations assume the form

$$\ddot{x}-k_1\dot{y}-n_1^2x=\mu f(x,\dot{x},y,\dot{y}),$$
$$\ddot{y}+k_2\dot{x}-n_2^2y=\mu g(x,\dot{x},y,\dot{y})+Q\sin t.$$

One looks for a solution

$$x=a\sin(\omega_1 t+\varphi_1)+b\sin(\omega_2 t+\varphi_2)+d\cos t,$$
$$y=(ak_1+a_2)\cos(\omega_1 t+\varphi_1-\gamma_1)+b(bk_2+b_n)\cos(\omega_2 t+\varphi_2-\gamma_2)+c\sin t.$$

For $a_2=b_2=\gamma_1=\gamma_2=0$, and $a,\cdots$, all constant this is the solution for $\mu=0$. For $\mu\neq 0$ but small, it is assumed that c, d are the same as when $\mu=0$, and $a, b, \varphi_1, \varphi_2$ are slowly varying parameters with a_2, b_2, γ_1, γ_2 corrective terms of order μ. The approximate treatment of the system generalizes the well-known methods characteristic, for example, of the work of Kryloff and Bogoliuboff [see their Introduction to Non-Linear Mechanics, Princeton University Press, 1943; these Rev. **4**, 142]. The treatment is applied to forced oscillations of a mechanical system (monorail car) with gyroscopic stabilizer. [See N. V. Butenin, Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **10**, 1283–1292 (1940).] *S. Lefschetz* (Princeton, N. J.).

Četaev, N. G. Concerning the stability and instability of irregular systems. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 639–642 (1948). (Russian)

The author considers the perturbed system

$$(*)\qquad dx_s/dt=p_{s1}x_1+\cdots+p_{sn}x_n+X_s,\quad s=1,\cdots,n,$$

where the coefficients p_{sr} are real continuous bounded functions of t, and the perturbations X_s are holomorphic with respect to the $x_1,\cdots,x_n$ with coefficients which are real continuous and bounded in t. Also, the X_s are assumed to have no constant or first degree terms in the $x_1,\cdots,x_n$. The following result due to Liapounoff is proved [refer to Problème général de la stabilité du mouvement, Ann. of Math. Studies, no. 17, Princeton, 1947 [these Rev. **9**, 34], remarks on pp. 251, 254; σ is a positive number defined on p. 251]. If the system of differential equations of the first approximation to (*) is irregular, and if all its characteristic numbers are larger than σ, then the unperturbed motion is stable. A second result is given: if the first approximation system is irregular, and if the least characteristic number is less than $-\sigma$, then the unperturbed motion is unstable. *E. A. Coddington* (Cambridge, Mass.).

Arf, Cahit. Sur l'existence de la solution d'un problème d'élasticité. Rev. Fac. Sci. Univ. Istanbul (A) **14**, 75–85 (1949). (French. Turkish summary)

The author considers a certain system of nonlinear equations for real numbers $\beta_1,\cdots,\beta_{n-1}$ which arose out of a problem in plane elasticity discussed previously by him [same Rev. (A) **12**, 309–344 (1947); these Rev. **10**, 217]. It is stated that the system (7.12) of the previous paper is inexact, and the actual system of equations is given. This nonlinear system is shown to have a solution for any integer $n\geqq 2$. Examination of the cases $n=2$ and $n=3$ reveals that the solution is unique then, but an example is given, for $n=4$, which has more than one solution. *J. B. Diaz.*

Avakumović, Vojislav G. Sur le problème aux limites des équations différentielles du second ordre non linéaires. Glas Srpske Akad. Nauka **191**, 53–66 (1948). (Serbian. French summary)

The author improves a result of A. Rosenblatt [Bull. Soc. Math. Grèce **14**, no. 1, 7–15 (1933)] concerning the boundary problem $y''=f(x,y,y')$, $y(a)=A$, $y(b)=B$. Making a linear transformation of the independent and dependent variables we can take $a=0$, $b=1$, $A=B=0$ and the author's

result then is the following. A solution of $y''=f(x, y, y')$, $y(0)=0$, $y(1)=0$, exists if:

$$(1)\quad |f(x, y_1, y_1')-f(x, y_2, y_2')| < \frac{\alpha|y_1-y_2|}{x^2(1-x)^2}+\frac{\beta|y_1'-y_2'|}{x(1-x)},$$

where $2\alpha+\beta<\frac{1}{2}$; and (2) there exists an integrable function $F(x)$ over $(0, 1)$ such that $|f(x, y, y')|<F(x)$ for $|y|<\Phi(x)$ and $|y'|<\Phi^*(x)$, where

$$\Phi(x)=(1-x)\int_0^x zF(z)dz+x\int_x^1(1-z)F(z)dz,$$

$$\Phi^*(x)=\int_0^x zF(z)dx+\int_x^1(1-z)F(z)dx.$$

A result on the minimum spacing of successive zeros is also given. *N. Levinson* (Cambridge, Mass.).

Avakumović, Vojislav G. Sur l'équation différentielle de Thomas-Fermi. Théorèmes relatifs à l'existence des intégrales. Glas Srpske Akad. Nauka **191**, 163–187 (1948). (Serbian. French summary)

The author obtains sufficient conditions on $F(x, y)$ for the existence of a solution $y(x)$ of $y''=F(x, y)$ where $y(0)=1$, $y(x)\to 0$ as $x\to\infty$. For the case $F(x, y)=f(x)y^\lambda$, $\lambda>0$, his sufficient condition is $f(x)$ continuous and $f(x)>0$ for $x>0$, $x^\alpha f(x)<\infty$ for small $x>0$ and $x^\alpha f(x)>c>0$ for large x for some $\alpha<2$. In this case the solution is unique.

N. Levinson (Cambridge, Mass.).

Akilov, G. P. On the application of a method of solution of nonlinear functional equations to the investigation of systems of differential equations. Doklady Akad. Nauk SSSR (N.S.) **68**, 645–648 (1949). (Russian)

Kantorovič has investigated the solution of an equation $P(x)=0$, where P is a nonlinear operator mapping into one another two spaces of type B [same Doklady (N.S.) **59**, 1237–1240 (1948); Uspehi Matem. Nauk (N.S.) **3**, no. 6(28), 89–185 (1948); these Rev. **9**, 537; **10**, 380]. His methods and results based essentially upon a method of successive approximation are applied here to the solution of a system over $0\leqq t\leqq 1$:

$$dx_s/dt=F_s(t, x_s, \cdots, x_n), \qquad s=0, 1, \cdots, n,$$

with boundary conditions

$$\Phi_s(x_0^0, \cdots, x_n^0;\ x_0^1, \cdots, x_n^1)=0,$$
$$x_i^0=x_i(0),\quad x_i^1=x_i(1),$$

where F_s and Φ_s are of class two in the variables other than t for t in $[0, 1]$. Periodic solutions are also discussed. Extensions are made to the infinite t-interval $[0, \infty]$ with applications to Liapounoff stability and also to the quasi-linear problems considered by Kryloff and Bogoliuboff [Introduction to Non-linear Mechanics; partial translation as Annals of Mathematics Studies, no. 11, Princeton University Press, Princeton, N. J., 1943; these Rev. **4**, 142]. The results are described without proofs. *S. Lefschetz* (Princeton, N. J.).

Satō, Tokui. On the limit of functional equations. Mem. Fac. Sci. Kyūsyū Univ. A. **4**, 23–27 (1949). (Esperanto)

Under various hypotheses concerning the behavior of the sequences $(\underline{D}^+f_n(x))$, $(\bar{D}^+f_n(x))$, $(f_n'(x))$, the convergence of the sequence $(f_n(x))$ to the solution $f'(x)=g(x)$ is established. The results are extended to cover an equation of the $f^{(k)}(x)=g(x)$. *R. Bellman* (Stanford University, Calif.).

Goldoni, Gino. Una dimostrazione del teorema di esistenza dell'integrale di una equazione differenziale del 1.° ordine in forma normale. Atti Soc. Nat. Mat. Modena (6) **26(79)**, 18–19 (1948).

In the square $|x-x_0|\leqq a$, $|y-y_0|\leqq a$ let $P_n(x, y)$ be a sequence of polynomials converging to the continuous function $f(x, y)$. Let y_n satisfy the differential system $y_n'=P_n(x, y_n)$, $y_n(x_0)=y_0$. Then $y=\lim_{n\to\infty} y_n$ satisfies $y'=f(x, y)$ on an interval $|x-x_0|\leqq b$, $0<b$.

J. M. Thomas (Durham, N. C.).

Cafiero, Federico. Sui teoremi di unicità relativi ad un'equazione differenziale ordinaria del primo ordine. II. Giorn. Mat. Battaglini (4) **2(78)**, 193–215 (1949).

This paper gives a theorem which modifies numerous details in the rather complicated hypothesis and preserves the conclusion of the principal theorem in a previous article [same vol., 10–41 (1948); these Rev. **10**, 457]. The object is to make the new theorem include the uniqueness criteria of G. Zwirner [Ann. Mat. Pura Appl. (4) **24**, 153–156 (1945); these Rev. **9**, 34] and G. Scorza-Dragoni [Rend. Circ. Mat. Palermo **54**, 430–448 (1930)]. *J. M. Thomas.*

Cafiero, Federico. Sui problemi ai limiti relativi ad un' equazione differenziale ordinaria del primo ordine e dipendente da un parametro. Rend. Sem. Mat. Univ. Padova **18**, 239–257 (1949).

For a boundary problem of the form

$$y'(x)=f(x, y(x), \lambda),\quad x_1\leqq x\leqq x_2,$$
$$y(x_1)=\varphi(\lambda),\quad \psi(y(x_2), \lambda)=0$$

the author establishes various conditions for the existence of a solution, using, in particular, semi-continuity properties with respect to λ of the upper and lower solutions of the differential equation. The problem includes as special cases those studied earlier by Takahashi [Tôhoku Math. J. **34**, 249–256 (1931)], Zawischa [Monatsh. Math. Phys. **37**, 103–124 (1930)] and the author [Giorn. Mat. Battaglini (4) **1(77)**, 145–163 (1947); these Rev. **10**, 194]. The paper concludes with some uniqueness theorems.

W. T. Reid (Evanston, Ill.).

Ważewski, Tadeusz. Sur les intégrales d'un système d'équations différentielles tangentes aux hyperplans caractéristiques issue du point singulier. Ann. Soc. Polon. Math. **21** (1948), 277–297 (1949).

On trouvera dans ce travail un théorème très général relatif aux solutions du système $du_i/dt=\varphi^i(u_1, \cdots, u_n)$ (où $i=1, \cdots, n$) quand on suppose les φ nulles pour $u_1=\cdots=u_n=0$, continues au voisinage de ce point (Θ), et possédant la différentielle de Stoltz en (Θ). Pour donner une idée de ce théorème dans un cas particulier, supposons que les racines caractéristiques de la matrice $\|\varphi_{u_j}^i\|$ soient réelles et distinctes: $s_1<\cdots<s_n$, et désignons par $d_1, \cdots, d_n$ les "droites caractéristiques" correspondant respectivement à ces racines; si pour un certain i on a $s_i<0$, alors la totalité des intégrales qui pour $t=+\infty$ sont en Θ tangentes à d_i se projette sur le plan passant par $d_1, \cdots, d_i$ comme un ensemble à i dimensions. Le théorème général est démontré en utilisant un hypothèse d'unicité des solutions correspondant à certaines données; mais on peut se passer de cette hypothèse en recourant à un principe topologique indiqué par l'auteur dans deux publications antérieures [mêmes Ann. **20** (1947), 279–313 (1948); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 210–215 (1947); ces Rev. **10**, 122; **9**, 512]. *M. Janet* (Paris).

Ehresmann, Charles, et Libermann, Paulette. Sur le problème d'équivalence des formes différentielles extérieures quadratiques. C. R. Acad. Sci. Paris **229**, 697–698 (1949).

Let Ω denote a quadratic differential form which involves $2n$ variables and has rank $2n$ under Grassmann multiplication of the differentials. The present note defines the adjoint Π^* of a form Π and the codifferential $\delta\Pi$ of Π with respect to Ω, the latter being $(d(\Pi^*))^*$, where d denotes taking the derived form or differential. Identities among powers, differentials, codifferentials and adjoints are given and applied to the problem of equivalence previously considered by the authors [same C. R. **227**, 420–421 (1948); these Rev. **10**, 122]. It is remarked that the operator $d\delta+\delta d$ can be applied to generate invariants useful in putting Ω into canonical form. *J. M. Thomas* (Durham, N. C.).

Chiellini, Armando. La teoria invariantiva del sistema differenziale formato da due equazioni lineari di ordine qualunque. I. Pont. Acad. Sci. Acta **12**, 199–213 (1948).

Chiellini, Armando. La teoria invariantiva del sistema differenziale formato da due equazioni lineari di ordine qualunque. II. I sistemi riducibili a coefficienti costanti. Pont. Acad. Sci. Acta **12**, 279–292 (1948).

Consider a system

$$\text{(1a)}\qquad \frac{d^n Y_\mu}{dx^n}+\sum_{\nu=1}^{2}\sum_{k=1}^{n}\Gamma_{\mu k}^{\nu}\frac{d^{n-k}Y_\nu}{dx^{n-k}}=0, \qquad \mu=\text{I, II},$$

with

$$\text{(1b)}\qquad \Gamma_{\mu 1}^{\mu}=0 \qquad \text{(not summing for } \mu).$$

The transformation

$$\text{(2a)}\qquad {}^*Y_\mu=Y_\mu\alpha(x), \quad {}^*x=\varphi(x)$$

with

$$\text{(2b)}\qquad \frac{d\log\alpha}{dx}+\frac{n-1}{2}\eta=0, \qquad \eta\equiv\frac{d}{dx}\log\frac{d\varphi}{dx},$$

carries (1a) into a similar system, which (after being divided by $\alpha(d\varphi/dx)^n$) will be denoted by *(1a), and (1b) into

$$\text{*(1)}\qquad {}^*\Gamma_{\mu 1}^{\mu}=0 \qquad \text{(not summing for } \mu).$$

[A restriction similar to (2b) imposed on (2a) is often used in geometrical applications. Cf., for instance, W. Stakowski, Arch. Math. **1**, 200–204 (1948); these Rev. **10**, 402.]

Comparing (1a) and *(1a) one obtains the transformations ${}^*\Gamma_{\mu k}^{\mu}\to\Gamma_{\mu r}^{\nu}$ (expressed by means of η) which split into two groups

$$\text{(3)}\qquad \text{(a)}\ {}^*\Gamma_{\mu k}^{\mu}\to\Gamma_{\mu r}^{\mu}, \qquad \text{(b)}\ {}^*\Gamma_{\mu k}^{\nu}\to\Gamma_{\mu r}^{\nu}\ (\mu\neq\nu)$$

(not summing for μ). The first group yields at once a set of $2(n-2)$ relative invariants $\theta_u^{(\omega)}$ $(u=3,\cdots,n)$ of weight u and two couples of relative invariants $\vartheta_v^{(\omega)}$ $(v=1,2)$ of weight v. [These invariants are closely related to the invariants found by the reviewer in Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **16**, 109–114, 206–211, 299–304 (1932).] Applying to (3b) the Schlesinger method of undetermined coefficients [cf. Schlesinger, Handbuch der Theorie der linearen Differentialgleichungen, v. 2, part 1, Teubner, Leipzig, § 181] one obtains another set of $2(n-2)$ relative invariants $\vartheta^{(\omega)}$. The invariants $\theta^{(\omega)}$, $\vartheta_v^{(\omega)}$, $\vartheta^{(\omega)}$ and $\vartheta_{1,1}^{(\omega)}$ (which one obtains by considering the transformation rule for $(d/dx)\vartheta_v^{(\omega)}$ and $(d^2/dx^2)\vartheta_v^{(\omega)}$) constitute a complete set of $2(2n-1)$ invariants which determines the $\Gamma_{\mu k}^{\nu}$.

On the other hand, if by some transformation (2) the system (1) can be reduced to *(1) with constant *Γ, then this transformation is defined by (2), where $\eta=(d/dx)\log\vartheta_1^{(1)}$. These results are illustrated by some examples (Fuchs's (pseudo)-systems of the first type and geometrical applications). *V. Hlavatý* (Bloomington, Ind.).

Germay, R. H. Remarque sur un théorème d'existence des solutions de certains systèmes d'équations aux dérivées partielles du premier ordre. Bull. Soc. Roy. Sci. Liége **18**, 127–133 (1949).

Démonstration, grâce à une fonction majorante convenable, d'un cas particulier du théorème fondamental de Cauchy, à savoir celui où le système donné s'écrit:

$$\frac{\partial u_i}{\partial x_1}=A_{kj}^{(i)}\frac{\partial u_k}{\partial x_j}+A^{(i)},$$

$$i,k=1,2,\cdots,p;\ j=2,3,\cdots,n,$$

les $A_{kj}^{(i)}$, $A^{(i)}$ étant des fonctions holomorphes données des seules inconnues u. (On passera très simplement de ce cas à celui où les $A_{kj}^{(i)}$, $A^{(i)}$ contiennent, de plus, les variables x). Le lemma analogue de "certains exposés classiques," dit l'auteur, suppose les $A^{(i)}$ identiquement nuls. *M. Janet.*

Martis-Biddau, Silvia. Sulla caratterizzazione di alcune classi di funzioni. Collectanea Math. **1**, 67–84 (1948).

Fantappiè proposed the problem of characterizing, in terms of systems of partial differential equations, the class of functions that can be obtained in one of the following forms: (a) $z=g_1(x_1)+\cdots+g_n(x_n)$, where $x_1,\cdots,x_n$ are independent variables; (b) $y=g(z)$, where z is of the form (a); (c) functions obtained from those of type (b) by replacing some or all of the independent variables by functions of type (a); (d) functions of sums of functions of type (c); (e) functions obtained by repeating these processes a finite number of times. It is understood that at each stage the new variables are independent from each other and from the old variables that are not replaced. It is also understood that all the functions used have the regularity properties required in the following proofs. The author shows that at each stage there exists a system of partial differential equations which characterizes the functions arising at that particular stage. As an example, consider functions of the type $z=g_1(u)+g_2(v)$, where $u=g_{11}(x_1)+g_{12}(x_2)$, $v=g_{21}(x_3)+g_{22}(x_4)$. Introduce the notations $D_{ik}=\partial^2/\partial x_i\partial x_k$, $L_{ik}=\log\partial/\partial x_i-\log\partial/\partial x_k$. Then a function $z(x_1,x_2,x_3,x_4)$ is of the indicated form if and only if it satisfies the following system: $D_{13}z=D_{14}z=D_{23}z=D_{24}z=0$, $D_{12}L_{13}z=D_{34}L_{34}z=0$. The author notes that her results lead to systems of partial differential equations that can be solved without quadratures. *T. Radó* (Columbus, Ohio).

Levinson, N., Bogert, B., and Redheffer, R. M. Separation of Laplace's equation. Quart. Appl. Math. **7**, 241–262 (1949).

The type of problem with which the present paper is concerned is to determine all coordinate systems in which a given equation can be solved by separation of variables. The emphasis here is on Laplace's equation. The discussion is confined to three dimensions. The following results are obtained. (I) For the Laplace equation $\Delta\Theta=0$ in curvilinear coordinates (u,v,w) in Euclidean space to be directly separable into two equations, one for S and one for Z, when

the solution is $\Theta=R(u, v, w)S(u, v)Z(w)$ with fixed R, it is necessary and sufficient that the surfaces $w=$constant (1) are orthogonal to the surfaces $u=$constant, $v=$constant and (2) are parallel planes with a common axis, concentric spheres, spheres tangent at a common point, or one of the two sets of spheres generated by the coordinate lines when bicircular coordinates are rotated about the line joining the poles or about its perpendicular bisector. (II) In the first three cases $R=1$ when and only when the surfaces $w=$constant are parallel planes, planes with a common axis, or concentric spheres. (III) In these three cases, and these only, the wave equation separates in the sense RSZ and hence, for the wave equation, $R=1$ automatically. (IV) For further separation of the equation found above for S, when $S=X(u)Y(v)$, so that the solution is now $RXYZ$, it is necessary and sufficient that the coordinates be toroidal, or such that the wave equation so separates, or any inversions of these. (V) The coordinates where the wave equation so separates, that is, admits solutions $RX(u)Y(v)Z(w)$, are only the well-known cases where this happens with $R=1$, namely, degenerate ellipsoidal or paraboloidal coordinates. (VI) In these cases, but only these, $R=1$ for the Laplace equation too. (VII) Coordinates for RSZ or $RXYZ$ separation of the Laplace equation have the group property under inversion. (VIII) In all cases R can be found by inspection of the linear element. *M. Pinl* (Dacca).

Šerman, D. I. On the theory of steady oscillations of a medium under given exterior forces on its boundary. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 557–560 (1949). (Russian)

The author considers the steady oscillation of an elastic medium which lies in a simply connected plane region S bounded by a "sufficiently smooth" curve L. The exterior forces acting on the medium are given on L. The longitudinal and transverse potentials, $\phi(x, y)$ and $\psi(x, y)$, respectively, must satisfy the equations $a^2\Delta\phi+\gamma^2\phi=0$, and $b^2\Delta\psi+\gamma^2\psi=0$, where a and b are the velocities of propagation of the longitudinal and transverse waves respectively, Δ is the Laplacian, while γ is the frequency of the oscillation. The problem consists in finding functions $\phi(x, y)$ and $\psi(x, y)$ satisfying the above equations in S and also the imposed boundary conditions along the curve L. This problem is reduced to the solution of a pair of integral equations of the Fredholm type. The author states that the existence proof for the solution of this system for almost all values of γ is analogous to that given in an earlier paper [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 259–266 (1947); these Rev. **9**, 121]. *H. P. Thielman.*

Westphal, H. Zur Abschätzung der Lösungen nichtlinearer parabolischer Differentialgleichungen. Math. Z. **51**, 690–695 (1949).

With the help of simple estimates the author proves uniqueness theorems for solutions $u(x, t)$ of a nonlinear parabolic equation $u_t=f(x, t, u, u_x, u_{xx})$, where f is assumed to be a monotone increasing function of u_{xx}. It is proved that u is uniquely determined in the rectangle $0<x<1$, $0<t<T$ in the (x, t)-plane if the values of u are prescribed on the two vertical and the lower horizontal side of the rectangle, provided f satisfies a Lipschitz condition with respect to the argument u. In this connection the Lipschitz condition can be replaced by weaker conditions, but an example shows that monotonicity in u_{xx} alone is not sufficient to guarantee uniqueness of the solution of the boundary value problem. *F. John* (New York, N. Y.).

Laasonen, Pentti. Über eine Methode zur Lösung der Wärmeleitungsgleichung. Acta Math. **81**, 309–317 (1949).

The author solves the parabolic equation

$$\kappa^2u_{xx}-u_t+q(x, t)=0$$

$(0\leqq x\leqq L, 0\leqq t\leqq T)$ subject to the boundary conditions

$$u(x, 0)=f_0(x), \quad u(0, t)=f_1(t), \quad u(L, t)=f_2(t),$$

where q_{xx}, q_t, $f_0^{(4)}$, f_1'', f_2'' are assumed to exist. He finds u by means of a modification of the method applied by R. Courant, K. Friedrichs, and H. Lewy to the homogeneous heat equation on an infinite interval [Math. Ann. **100**, 32–74 (1928)], in which u is represented as limit of solutions u_n of difference equations belonging to a rectangular lattice. Using a different form of the difference quotient the author establishes the convergence of a suitable subsequence of the u_n towards a solution u of the original problem, as the sides h and k of the fundamental rectangle of the lattice tend to zero in any manner. In contrast to the corresponding situation for hyperbolic equations no inequalities between h and k are needed here to assure convergence. *F. John.*

Boyarincev, D. I. The heating of a cylinder by solar radiation. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 1469–1480 (1949). (Russian)

The author considers the heating of cylindrical tanks with hemispherical ends, and of a pipeline, due to direct and reflected solar radiation. First, through spherical trigonometry, the rate Q of heat gain is found. Then the heat balance equation leads to a second order differential equation for the temperature difference between the contents of the tank and the surrounding temperature. A series solution of this equation is given for the case when Q is a function of time alone. Some numerical calculations are included.

R. E. Gaskell (Ames, Iowa).

Berlyand, O. S. On the theory of the distribution of temperature in the Earth's atmosphere. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **13**, 388–392 (1949). (Russian)

The differential equation in question,

$$d^2y/d\tau^2-(1+\phi)y=-\phi e^{-\beta\tau},$$

is solved by I. A. Kibel [C. R. (Doklady) Acad. Sci. URSS (N.S.) **39**, 17–21 (1943)] only for $\phi=M=$constant. The present article solves via series the same equation when $\phi=M$ $(1\leqq\tau\leqq12.6)$, $\phi=M/\tau$ $(0\leqq\tau\leqq1)$. With the physical constants chosen, $\tau=1$ corresponds to the upper limit of the troposphere (about 10.5 km.) while the range $1\geqq\tau>0$ corresponds to the stratosphere (above 10.5 km.). Calculations given for the temperature up to 13 km. were little different from results of Kibel. *R. E. Gaskell.*

Schunack, Johannes. Die Erwärmung eines inhomogenen Stabes und die Ladung eines inhomogenen Thomson-Kabels. Arch. Elektr. Übertragung **2**, 190–207 (1948).

Through the face $x=0$ of a uniform solid $0\leqq x\leqq L$ there is a prescribed constant flux of heat, a situation that is approximated by the heat generated during the impact of a steady stream of electrons upon the face of a metallic anode. The face $x=L$ is insulated and the initial temperature is uniform. Operational calculus is used to write the known formulas for the temperatures $U(x, t)$ in the solid for the two cases in which L is infinite and finite, and also in the first case when the body is composite with one layer

$0 \leqq x \leqq a < L$ of a different material from the rest. A graphical study of the temperature functions is presented. One of the electrical interpretations of these mathematical problems is described: $U(x, t)$ represents the electric potential and the flux of heat corresponds to the current in a transmission line whose inductance per unit length can be considered as zero. A table of corresponding parameters in the two physical problems is given. *R. V. Churchill.*

Bureau, Florent. Sur l'intégration d'une équation linéaire aux dérivées partielles totalement hyperbolique d'ordre quatre et à trois variables indépendantes. Acad. Roy. Belgique. Cl. Sci. Mém. Coll. in 8°. (2) **21**, no. 6, 64 pp. (1948).

The equation considered is of the type

$$(1) \qquad f(\partial/\partial x_1, \cdots, \partial/\partial x_p)u = 0,$$

where f is a form of degree n with constant coefficients. Previously the author [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 587–610, 684–711, 827–853 (1947); these Rev. **9**, 513; **10**, 855] has derived expressions for the elementary solution F of (1) in terms of integrals over the $(p-2)$-dimensional algebraic manifold $f(\alpha_1, \cdots, \alpha_{p-1}, 1) = 0$. A series of his papers has been devoted to the explicit evaluation of these integrals in special cases and to the corresponding formulae for the solution of the Cauchy problem [see C. R. Acad. Sci. Paris **222**, 849–851 (1946); **225**, 402–403, 852–854 (1947); **226**, 150–152, 1331–1333 (1948); Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 379–402, 473–484 (1947); **34**, 566–592 (1948); these Rev. **7**, 450; **9**, 188, 441, 591, 356, 441; **10**, 459]. The present paper contains a detailed discussion of the case where f is of the form

$$f(\alpha_1, \alpha_2, \alpha_3) = (a_1\alpha_1^2 + a_2\alpha_2^2 + a_3\alpha_3^2)(b_1\alpha_1^2 + b_2\alpha_2^2 + b_3\alpha_3^2).$$

It is found that in this case F can be expressed explicitly in terms of elementary functions. If all a_i, b_i are positive, equation (1) is elliptic and F is the fundamental solution of (1). If a_3, b_3 are positive, and a_1, b_1, a_2, b_2 negative, we have a totally hyperbolic equation with a characteristic cone consisting of two quadratic cones Γ^{I} and Γ^{II}. In this case F is given by different expressions in the different regions bounded by these cones or their convex hull. With the help of F the author solves the Cauchy problem for (1) with Cauchy data prescribed on an arbitrary spacelike surface. In the special case, where the initial surface is the (x_1, x_2)-plane and the quadratic cones do not intersect, the resulting formula for the solution of the Cauchy problem reduces to the one given by G. Herglotz [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **80**, 69–114 (1928)]. *F. John* (New York, N. Y.).

Calculus of Variations

Giuliano, Landolino. Sulla continuità degli integrali curvilinei del calcolo delle variazioni. Ann. Scuola Norm. Super. Pisa (3) **1** (1947), 161–187 (1949).

The author has previously shown that the integral $\int(M + Ny' + Pz')dx$ is a continuous function of the path for absolutely continuous curves $y = y(x)$, $z = z(x)$ interior to a region A of (x, y, z)-space where $M, N, P, N_z, P_z, N_z, P_y$ are continuous functions of x, y, z, satisfying $N_z = P_y$ [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 39–45 (1948); these Rev. **10**, 259]. The present paper extends the result to the case when the curves are allowed to lie on the boundary of A, subject to a number of restrictions on A and on the curves. *L. M. Graves* (Chicago, Ill.).

La Vallée Poussin. Sur les extrémales d'une certaine intégrale I et l'appoint qu'elles apportent à un théorème général d'unicité concernant l'intégrale d'une equation différentielle linéaire. Pont. Acad. Sci. Acta **12**, 141–159 (1948).

The author considers the extremals of the integral $I(C) = \int_C r\,ds$, where r is the radius vector in the (x, y)-plane, and s is the length of arc along the curve C. The obtained results lead readily to a recent uniqueness theorem of Ballieu [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 725–742 (1947); these Rev. **9**, 435] for linear differential equations $w^{(n)} + \chi_1(z)w^{(n-1)} + \cdots + \chi_n(z)w = 0$, where $\chi_1(z), \cdots, \chi_n(z)$ are holomorphic functions of the complex variable z in a convex domain. *W. T. Reid* (Evanston, Ill.).

McFarlan, L. H. The transversality condition of the calculus of variations. Univ. Washington Publ. Math. **3**, no. 1, 15–20 (1948).

Given a function $f(x, y, p)$ then, in the terminology of the calculus of variations, q is transversal to p at (x, y) if $f(x, y, p) + (q - p)f_p(x, y, p) = 0$. The present paper is concerned with conditions under which this equation has a unique solution $p = T(x, y, q)$ and under which $p = T(x, y, q)$ if and only if $q = T(x, y, p)$. In this latter case the transversality condition is involutoric. It is shown that if the transversality condition is of the form

$$A(x, y, p)[P(x, y)pq + Q(x, y)(p + q) + R(x, y)] = 0$$

then it is involutoric and f is of the form $f = c(Pp^2 + 2Qp + R)^{\frac{1}{2}}$. This result is generalized to higher dimensions.

M. R. Hestenes (Los Angeles, Calif.).

Bilimovitch, Anton D. Hilbert's integral of independence and Pfaff's equations of calculus of variations. Glas Srpske Akad. Nauka **189**, 169–180 (1946). (Serbian. English summary)

The author points out that, if the integrand of the Hilbert invariant integral is regarded as a Pfaffian form, the equations of Pfaff lead to the Euler equations. The manipulations involved are simple and rather well known.

L. M. Graves (Chicago, Ill.).

Wagner, V. Theory of field of local $n-2$-dimensional surfaces in X_n and its application to the problem of Lagrange in the calculus of variations. Ann. of Math. (2) **49**, 141–188 (1948).

The problem of Lagrange is to find the extremum of $s = \int_{t_1}^{t_2} F(\xi^\alpha, \dot{\xi}^\alpha)dt$ under the condition $\Phi(\xi^\alpha, \dot{\xi}^\alpha) = 0$, where F and Φ satisfy the usual homogeneity and nondegeneracy assumptions. The essential idea of this paper is to interpret the problem as a theory of local $(n-2)$-dimensional surfaces in the space X_n, one such surface in each tangent space of X_n, and to solve the problem of equivalence of two problems of Lagrange on the basis of this geometrical interpretation. Part I deals with the centro-affine geometry of an $(n-2)$-dimensional surface in an n-dimensional affine space. Differential invariants are found which determine the surface up to a centro-affine transformation. Various geometrical interpretations are given to these differential invariants. Part II gives the definition of an absolute differential in a compound manifold, the space of a field of $(n-2)$-dimensional

surfaces. A complete set of differential invariants is given, and some of them are geometrically interpreted. For instance, conditions are given that the field be flat. In part III the geometrical theory is applied to the study of the problem of Lagrange as a problem of the calculus of variations. In particular, the first variation and the equations of the extremals are determined and a sufficiency condition is proved. *S. S. Chern* (Chicago, Ill.).

Reid, William T. Expansion methods for the isoperimetric problem of Bolza in non-parametric form. Amer. J. Math. **71**, 946–975 (1949).

During the last few years three methods for establishing sufficiency theorems for the problem of Bolza have been devised, namely, the method of Mayer fields, the expansion method and the indirect method. In the case when isoperimetric side conditions are adjoined, heretofore a transformation of the problem appeared to be necessary except in the case of the indirect method. In the present paper it is shown that the expansion method can be extended so as to be applicable to isoperimetric problems without transforming the problem. These results are obtained without the use of the theorem of Lindeberg. The paper concludes with a study of the properties of the second variation for isoperimetric and nonisoperimetric problems. *M. R. Hestenes.*

Chen, Yu Why. Existence of minimal surfaces with a simple pole at infinity and condition of transversality on the surface of a cylinder. Trans. Amer. Math. Soc. **65**, 331–347 (1949).

Let C be a simple closed smooth curve in the (x, y)-plane. The problem of determining a circulation-free flow of a gas past C which has at infinity a given velocity reduces to a boundary value problem for minimal surfaces if the gas is assumed to satisfy Chaplygin's simplified equation of state $(\gamma=-1)$. This is the author's problem I: to find a minimal surface $z=z(x, y)$ defined in the domain exterior to C and satisfying the conditions: $\partial z/\partial n=0$ on C; $\partial z/\partial x\to\beta$, $\partial z/\partial y\to 0$ as $x^2+y^2\to+\infty$. It is shown that a solution admits the parametric representation

$$(\dagger)\quad \begin{aligned} x(u, v) &= -a(1+\beta^2)^{-\frac{1}{2}}(r^{-1}-r)\cos\theta+X(u, v),\\ y(u, v) &= a(r^{-1}-r)\sin\theta+Y(u, v),\\ z(u, v) &= -\beta a(1+\beta^2)^{-\frac{1}{2}}(r^{-1}-r)\cos\theta+\text{constant},\end{aligned}$$

$|u+iv|=|re^{i\theta}|<1$, where X, Y are harmonic functions mapping (monotonically) $u^2+v^2=1$ onto C and a is a constant such that (†) represents a minimal surface. Problem II consists of finding such X, Y, a. The author shows that problem II possesses a solution. This is accomplished by direct methods of calculus of variations, by minimizing the difference between the Dirichlet integral of (X, Y) and a certain line integral, only functions possessing the required mapping property being admitted to competition. It is shown that the solution of problem II depends continuously upon β. The author states, however, that he has not yet been able to show that the solution of problem II actually solves problem I. [The reviewer obtained an existence theorem for a boundary value problem closely related to though not identical with problem I [Proc. Symposia Appl. Math., vol. I, pp. 41–46, Amer. Math. Soc., New York, 1949; these Rev. **10**, 753]. For sufficiently small values of β the solvability of problem I follows from a result by Frankl and Keldysch [Bull. Acad. Sci. URSS (7) Cl. Sci. Math. Nat. [Izvestiya Akad. Nauk SSSR] **1934**, 561–601].] *L. Bers* (Princeton, N. J.).

Theory of Probability

Cernuschi, Félix. A noncontradictory formulation of the collective of von Mises. Ciencia e Investigación **5**, 258–260 (1949). (Spanish)

As usual, the possible selections of subsequences are mapped onto the points of (0, 1) using the dyadic representation of numbers. The second axiom of von Mises is then replaced by the requirement that with probability one a selection satisfies the von Mises axiom. [It is not clear to the reviewer how the probability occurring in this statement is to be defined.] *W. Feller* (Ithaca, N. Y.).

Franckx, E. Sur les probabilités d'arrivée des événements en nombre infini. Skand. Aktuarietidskr. **32**, 7–14 (1949).

The author considers infinite sequences of events with known probabilities. He establishes by elementary methods inequalities for the probabilities that a finite or an infinite number of events of the sequence are or are not realized. These inequalities are used to give elementary proofs of a number of known theorems concerning infinite sequences of random variables. All these theorems state properties which are almost certain. In the reviewer's opinion the formula $p\leqq\lim_{k\to\infty} p_{A_k}\leqq P$ on the first page of the paper should be replaced by $p\leqq P$ as the existence of $\lim_{k\to\infty} p_{A_k}$ is not used in the proof. On the other hand, theorem E of the author assumes that this limit does not exist. There are many misprints in the paper. *E. Lukacs* (Washington, D. C.).

Seal, H. L. The historical development of the use of generating functions in probability theory. Mitt. Verein. Schweiz. Versich.-Math. **49**, 209–228 (1949).

Borel, Émile. Sur une martingale mineure. C. R. Acad. Sci. Paris **229**, 1181–1183 (1949).

In a coin tossing game with arbitrary states the gambler adopts the rule of multiplying after each trial the preceding stake by $1+\alpha$ or $1-\alpha$, according as the last trial resulted in success or failure $(0<\alpha<1)$. Let the initial state be a and suppose that the first $p+q$ trials resulted in p successes and q failures. The author shows that then the accumulated gain is $\{a-(1+\alpha)^q(1-\alpha)^p\}/\alpha$. Here p and q are random variables, but the author concludes that since $p/q\to 1$ with probability one, $(1+\alpha)^q(1-\alpha)^p\to 0$. This would imply that the gain tends to a/α. Furthermore, the author claims that the stakes remain bounded. For practical gambling he recommends choosing α between $\frac{1}{3}$ and $\frac{1}{4}$. *W. Feller.*

Moran, P. A. P. Numerical integration by systematic sampling. Proc. Cambridge Philos. Soc. **46**, 111–115 (1950).

Let $f(x)$, $-\infty<x<\infty$, be of bounded variation and Lebesgue integrable. Define $S(t)=\delta\sum_{n=-\infty}^{\infty} f(t+s\delta)$, and suppose that t is a random variable uniformly distributed in the interval $(0, \delta)$. It is proved that $S(t)$ is then a random variable with mean $\int_{-\infty}^{\infty} f(x)dx$, and the variance is calculated. The variance is evaluated in several examples. Corresponding results are outlined in the two variable case $f=f(x, y)$. *J. L. Doob* (Ithaca, N. Y.).

★Darmois, G. Sur certaines formes de liaisons de probabilité. Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 19–21. Centre National de la Recherche Scientifique, Paris, 1949.

Let x, y be random variables with distribution functions A, B and bivariate distribution function F. The author

gives classes of examples in which $F-AB$ never changes sign. Lévy shows in an appended note that for every given pair A, B, there is an F for which $F-AB$ does not change sign. *J. L. Doob* (Ithaca, N. Y.).

Kunisawa, Kiyonori. On an analytical method in the theory of independent random variables. Ann. Inst. Statist. Math., Tokyo 1, 1–77 (1949).

This paper consists of six parts of which the first two develop what may be called the method of mean concentration functions. In the remaining four parts this method is used to derive by unified methods various theorems which play an important role in the theory of summation of independent random variables. Let $F(x)$ be a distribution function, $F^*(x)=F(x)*\{1-F(-x)\}$ the corresponding symmetrized distribution, $f(t)$ the characteristic function of $F(x)$. Furthermore, let $\mu(x)$ be an even density function, $m(t)$ its characteristic function. The generalized mean concentration function of $F(x)$ is defined by

$$(1)\quad g_F(h)=h(2\pi)^{-1}\int_{-\infty}^{\infty}\mu(ht)\,|f(t)|^2dt=\int_{-\infty}^{\infty}m(x/h)dF^*(x),$$

$(h>0)$. Putting $m(x)=1-|x/2|$ for $|x|<2$ and $m(0)=0$ for $|x|>2$ one gets the function introduced by Kawata [Duke Math. J. 8, 666–677 (1941); these Rev. 3, 168]. P. Lévy's concentration function $Q(h)=\sup\{F(x+h+0)-F(x-0)\}$ is not a special case, but the author shows that

$$g_F(h)\geqq m(a)Q^2(ah).$$

For his own purposes the author almost invariably puts $m(x)=(1+x^2)^{-1}$, $\mu(t)=(\pi/2)^{\frac{1}{2}}\exp(-|t|)$; the resulting $g_F(x)$ is denoted by $\Psi_F(h)$ and called the mean concentration function. The function

$$(2)\quad \Phi_F(l)=l\int_{-\infty}^{\infty}e^{-lt}Rf(t)dt=\int_{-\infty}^{\infty}l^2(l^2+x^2)^{-1}dF(x)$$

is called the typical function of $F(x)$.

The theory depends on a sequence of inequalities. It is easily seen that $1-\Psi_F(h)\leqq 2(1-\Phi_F(h))$. An inequality in the reverse direction is derived with the factor 2 replaced by $K(\alpha,\lambda)>0$, where α is an upper bound for $1-F(h)+F(-h)$ and $F(0)\geqq\lambda>0$, $F(-0)\leqq 1-\lambda<1$. Furthermore, if F is the convolution of $F_1,\cdots,F_n$, then $1-\Psi_F(h)\leqq\sum_1^n\{1-\Psi_{F_k}(h)\}$, and again an inequality in the reverse direction holds under certain restrictive conditions. Finally,

$$(3)\quad |f(t/h)\exp(-iat)-1|\leqq(l^2+2|t|+4)(1-\Phi_F(h)),$$

where $a=\int_{-1}^{1}xdF(hx)$. Perhaps the main result is the uniform diminution theorem to the effect that to every pair α,β in $(0,1)$ there exist positive numbers K and N such that for $n\geqq N$ the n inequalities $\Psi_{F_k}(l)\leqq\alpha$ imply $\Psi_F(n^{\frac{1}{2}}Kl)\leqq\beta$ (where $F=F_1*\cdots*F_n$). Similar theorems are obtained for the typical function and, more generally, for triangular arrays of distribution functions.

These results are used to derive the three series theorem and several results due to P. Lévy concerning the equivalence of various modes of convergence. Next necessary and sufficient conditions for the weak law of large numbers and various modifications are derived. In section 5 the author gives a new derivation of infinitely divisible distributions and their main properties, as well as necessary and sufficient conditions for the convergence towards such a function of a triangular array of distributions. Finally, the last chapter is devoted to the strong law of large numbers and to a slight extension of a Lévy-Marcinkiewicz limit theorem for sums of random variables for which the tails of the distribution behave uniformly like $x^{-\alpha}$ [the corresponding general form has been discovered by the reviewer in the meantime; cf. Amer. J. Math. 68, 257–262 (1946); these Rev. 8, 37]. *W. Feller* (Ithaca, N. Y.).

Bergström, Harald. On the central limit theorem in the case of not equally distributed random variables. Skand. Aktuarietidskr. 32, 37–62 (1949).

Let $X^{(p)}=(X_1^{(p)},\cdots,X_k^{(p)})$, $p=1,\cdots,n$, be k-dimensional random vectors with distribution functions $F_p(x)$, where $x=(x_1,\cdots,x_k)$. Let $F_n^*(x)=F_1(x)*\cdots*F_n(x)$ and let $\Phi_n(x)$ be the normal distribution function with the same mean and covariance matrix as $F_n^*(x)$. A true estimate of $|F_n^*(x)-\Phi_n(x)|$ for $k=1$ was first given by A. C. Berry and Esseen, independently. For $k>1$ and $F_p(x)=F(x)$ ("equal components") an extension was given by the author [Skand. Aktuarietidskr. 28, 106–127 (1945); these Rev. 7, 459]. In the present paper he treats the general case of unequal components for $k>1$. This case presents certain difficulties and the result becomes more complicated involving besides the second and third absolute moments also the ratios of certain minor determinants in the covariance matrices. The final estimate depends on k as a positive power of k. This is of importance in some applications where k is taken to be an increasing function of n; see a paper by the reviewer [Bull. Amer. Math. Soc. 54, 1162–1170 (1948); these Rev. 10, 384]. *K. L. Chung* (Ithaca, N. Y.).

Feller, William. Fluctuation theory of recurrent events. Trans. Amer. Math. Soc. 67, 98–119 (1949).

Let $X_1, X_2, \cdots$ be mutually independent positive integral valued random variables with a common distribution function. Let $\mu=E\{X_1\}$, $S_k=$ number of S_n's $\leqq k$. Then $X_1, X_2, \cdots$ can be identified as the recurrence times of a recurrent event E and S_k becomes the number of realizations of E in k trials. (In stating the following results it will be supposed for simplicity that 1 is the highest common factor of values of k with $\Pr\{X_1=k\}>0$ and that X_1 is finite valued.) If u_n is the probability that E occurs at the nth trial and if $\mu\leqq\infty$ then $\lim_{n\to\infty}u_n=1/\mu$. This theorem, proved by generating functions, can be deduced from a Markov chain theorem due to Kolmogorov [Rec. Math. [Mat. Sbornik] N.S. 1(43), 607–610 (1936)] and in turn can be used to deduce the asymptotic properties of the transition probabilities of a Markov chain with finitely or infinitely many states. The method of proof has been extended by Blackwell to prove the analogous theorem for continuous valued X_j's [Duke Math. J. 15, 145–150 (1948); these Rev. 9, 452]. Sharpening earlier work of Täcklind [Skand. Aktuarietidskr. 27, 1–15 (1944); 28, 68–105 (1945); these Rev. 7, 207] it is then shown that, if X_1 has a finite factorial second moment M, $\sum_0^n(u_j-1/\mu)=M/2\mu^2$ where the series on the left converges absolutely and has general term $o(1/j)$. The limiting distribution of N_n $(n\to\infty)$ is reduced to that of S_n and all possible limiting distributions of N_n are found in this way. An incidental result is an explicit power series for $\varphi(\lambda\Gamma(1-\alpha))$, where φ is the Laplace transform of the distribution function $1-G_\alpha(x^{-1/\alpha})$ and G_α is the distribution function of the stable law with exponent $\alpha<1$. This series was obtained directly by Pollard [Bull. Amer. Math. Soc. 54, 1115–1116 (1948); these Rev. 10, 295]. Finally the law of the iterated logarithm and its generalizations are applied to the X_j's to yield corresponding theorems for the N_j's.

For example, if F is the distribution function of X_1, and if $\mu=\infty$, let $\psi(x)$ be a function with $\psi(x)\uparrow$ and $\psi(x)/x\downarrow$; then, with probability 1, $N_n<\psi(n)$ will be true for infinitely many n or at most finitely many n according as $\int[1-F(x)]d\psi(x)$ diverges or converges. Applications are made to Markov chains, runs, equalizations of heads and tails in coin tossing. *J. L. Doob* (Ithaca, N. Y.).

Andersen, Erik Sparre. On the number of positive sums of random variables. Skand. Aktuarietidskr. **32**, 27–36 (1949).

An s-permutation $(x_1, \cdots, x_n)$ of n numbers $a_1, \cdots, a_n$ is a permutation $(\epsilon_{i_1}a_{i_1}, \cdots, \epsilon_{i_n}a_{i_n})$ of the n numbers $\epsilon_1a_1, \cdots, \epsilon_na_n$, where $\epsilon_i=\pm1$, $i=1, \cdots, n$. Let $X_1, X_2, \cdots$ be an infinite sequence of chance variables such that the first n $(n=1, 2, \cdots)$ have a density function which is invariant under all s-permutations of the arguments. Let $s_k=X_1+\cdots+X_k$ and let N_n denote the number of s_k, $1\leqq k\leqq n$, which are positive. It is proved that

$$\lim_{n\to\infty} P\{N_n/n\leqq\alpha\}=2\pi^{-1}\sin^{-1}\alpha^{\frac{1}{2}}, \qquad 0\leqq\alpha\leqq1.$$

This result partly overlaps, and is approximately simultaneous with, the results announced by Darling [Bull. Amer. Math. Soc. **55**, 703 (1949)]. *J. Wolfowitz.*

Špaček, Antonín. Note on successive cumulative sums of independent random variables. Časopis Pěst. Mat. Fys. **74**, 41–45 (1949). (English. Czech summary)

Let $z_1, z_2, \cdots$ be an infinite sequence of identically distributed chance variables with positive variances. Let $Z_j=\sum_{i=1}^{j}z_i$. Define the chance variable n as follows: $n=j$, where j is the smallest integer for which $Z_j\leqq b-jd$ or $a-jd\leqq Z_j\leqq b$, or $a\leqq Z_j$. Here $d>0$ and $b<0<a$ are constants. It is proved that n has moments of all orders; Wald's fundamental identity [Ann. Math. Statistics **15**, 283–296 (1944), equation (15); these Rev. **6**, 88] holds; and, speaking roughly, it is legitimate to differentiate the latter under the expectation sign. *J. Wolfowitz* (New York, N. Y.).

Sapogov, N. A. On a limit theorem. Doklady Akad. Nauk SSSR (N.S.) **69**, 15–18 (1949). (Russian)

Cramér and Wold [J. London Math. Soc. **11**, 290–294 (1936)] have shown how to find limiting multivariate distributions of vector random variables X_n by considering the one-dimensional random variables (U, X_n) for all fixed vectors U. According to the author the details of the method are simplified if one uses the following theorem. Let $\{X_n\}$, $\{Y_n\}$ be sequences of vector random variables. Suppose that $\lim_{N\to\infty}\Pr\{|Y_n|>N\}=0$ uniformly in n, that the distribution functions of the components of Y_n are continuous uniformly in n, and that

$$|\Pr\{(U, X_n)<\lambda\}-\Pr\{(U, Y_n)<\lambda\}|<\alpha_n$$

for all λ, U. Then $|\Pr\{X_n\epsilon I\}-\Pr\{Y_n\epsilon I\}|<\omega(\alpha_n)$ uniformly in the multidimensional interval I, where

$$\lim_{n\to\infty}\omega(\alpha_n)=0.$$

He gives as an example a proof of the central limit theorem for two-dimensional vector random variables (with the Y_n's the appropriate Gaussian variables). In this case at least it does not appear to the reviewer that any simplification has been achieved. *J. L. Doob* (Ithaca, N. Y.).

Sapogov, N. A. On multidimensional inhomogeneous Markov chains. Doklady Akad. Nauk SSSR (N.S.) **69**, 133–135 (1949). (Russian)

For each n let $X_1^{(n)}, \cdots, X_{k_n}^{(n)}$ be h-dimensional vector random variables; $X_i^{(n)}$ can take on $g_i^{(n)}$ possible values. It is assumed that there is a Markov chain with $g_i^{(n)}$ states at time t_i, and that when the chain is the jth state at time t_i, $X_i^{(n)}$ takes on its jth value. Linnik [Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 65–94 (1949); these Rev. **10**, 552] has given conditions sufficient that, in the case $h=1$ and $k_n=n$, the sum $\sum_i X_i^{(n)}$ be asymptotically normally distributed when $n\to\infty$. The author extends this to the general case, finding conditions that the multidimensional central limit theorem hold. The problem is reduced to the one-dimensional case by a method of Cramér and Wold, as expanded slightly by the author in the paper reviewed above. *J. L. Doob* (Ithaca, N. Y.).

Hille, Einar. Les probabilités continues en chaîne. C. R. Acad. Sci. Paris **230**, 34–35 (1950).

Let $a(x)$, $b(x)$ be continuous, with $b>0$, $-\infty<x<\infty$. Problem (C): Choose a, b so that $b(\xi)g''(\xi)+a(\xi)g'(\xi)$ is the infinitesimal operator of a uniquely determined semigroup $\{S(t), 0\leqq t<\infty\}$ of continuous (strong sense) linear operations on $g\epsilon C$, for which $\|S(t)\|=1$, $S(t)$ is positive, $S(t)1=1$, $S(0)=I$. Problem (L): Choose a, b so that $\{[b(x)h(x)]'-a(x)h(x)\}'$ is the infinitesimal operator of a uniquely determined semigroup $\{T(t), 0\leqq t<\infty\}$ of continuous (strong sense) linear operations on $h\epsilon L$ with $\|T(t)\|=1$, $T(t)$ positive, isometric on positive elements of L. Necessary and sufficient conditions too long to be reproduced here are found which solve these problems, with the (L) conditions stronger than the (C) conditions. If the (C) conditions are satisfied, the Green's function of the problem can be used to define the transition function of a Markov stochastic process with stationary transition probabilities, and thus find a solution of the diffusion (Kolmogorov-Fokker-Planck) equations. *J. L. Doob* (Ithaca, N. Y.).

Čulanovskiĭ, I. V. On cycles in Markov chains. Doklady Akad. Nauk SSSR (N.S.) **69**, 301–304 (1949). (Russian)

Let $E_1, \cdots, E_n$ be the states of a Markov chain. Any succession of states $E_{a_1}, \cdots, E_{a_m}$ (with no repetitions) with the property that transitions are possible from E_{a_j} to $E_{a_{j+1}}$ and from E_{a_m} to E_{a_1} determines an n-dimensional vector with ith component 1 or 0 according as E_i is or is not in the succession. It is proved that any n-dimensional vector with integral components which is a linear combination of the vectors obtained in this way is a linear combination of these vectors with integral coefficients. This verifies a conjecture of Kolmogorov [Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 281–300 (1949); these Rev. **11**, 119]. *J. L. Doob.*

Consael, R. Sur quelques processus stochastiques discontinus à deux variables aléatoires. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 743–755 (1949).

The author continues his study of the development of two species in accordance with various laws of growth [same Bull. Cl. Sci. (5) **34**, 863–876 (1948); these Rev. **10**, 552]. In the present paper he supposes that in time dt each individual of species 1 generates one of species 2 with probability $\lambda(t)dt$ whereas each individual of species 2 has probability $\lambda(t)dt$ of dying and simultaneously generating 2 individuals of species 1. It is supposed that initially there

is one individual of one species and none of the other, and the distribution of individuals at time t is found.

J. L. Doob (Ithaca, N. Y.).

Onicescu, O., et Mihoc, G. Chaînes de mouvements discontinus et changements d'état. Bull. Math. Soc. Roumaine Sci. **48**, 32–42 (1947).

The authors discuss a Markov process defined on the set of vertices of an infinite but locally finite network in which the transition probability P_{MN} of going from a vertex M to a vertex N is positive if and only if M and N are two neighboring vertices of the network. Various special cases when the network has a simple uniform structure are discussed. *S. Kakutani* (New Haven, Conn.).

Kakutani, Shizuo. Two-dimensional Brownian motion and the type problem of Riemann surfaces. Proc. Japan Acad. **21** (1945), 138–140 (1949).

Let Σ be a simply connected infinitely sheeted open Riemann surface. Let D be a simply connected finitely sheeted Jordan domain on Σ with at most a finite number of algebraic branch points on it, and let $u(z)$ be the probability that a Brownian motion with initial point $z \varepsilon \Sigma - \bar{D}$ will intersect the boundary of D before leaving Σ. Then if Σ is parabolic $u(z) \equiv 1$ and if Σ is hyperbolic $u(z) < 1$ for all z. (The problem is reduced to Σ the finite plane or the interior of a circle by conformal mapping.) *J. L. Doob.*

Hostinský, Bohuslav. Sur le calcul des probabilités relatives à l'évolution d'un système. Aktuárské Vědy **8**, no. 2, 61–67 (1949). (Czech. French summary)

The author recalls the work of E. Schoenbaum on a problem in probability theory which leads to an integro-differential equation of the Volterra type, and where each term of the Neumann series for the solution can be interpreted in terms of probabilities. After this, the author discusses the evolution of a system into which sudden changes are introduced. The actual problem treated is related to (without being identical with) one discussed in an earlier paper of the author [Rozpravy II. Třídy České Akad. **50**, no. 26 (1940); these Rev. **10**, 200]. Particular emphasis is put on the interpretation in terms of probabilities of the individual terms of the infinite series representing the solution. *A. Erdélyi* (Pasadena, Calif.).

Zwinggi, Ernst. Berechnung und Darstellung der abhängigen und unabhängigen Wahrscheinlichkeiten. Mitt. Verein. Schweiz. Versich.-Math. **49**, 179–193 (1949).

The author studies aggregates which decrease due to several causes. In the first part of the paper it is shown how the dependent as well as the independent annual rates of decrement can be obtained from the observed data. In the second part independent rates of decrement are expressed in terms of the dependent rates of decrement and vice versa. The paper supplements the second chapter (§§ 3 and 4, pp. 22–30) of a book by the same author [Versicherungsmathematik, Birkhäuser, Basel, 1945; these Rev. **8**, 57]. *E. Lukacs* (Washington, D. C.).

***Fraser, Donald A. S. Generalized Hit Probabilities with a Gaussian Target.** Princeton, N. J., 1949. ii+62 pp.

Cunningham and Hynd [Suppl. J. Roy. Statist. Soc. **8**, 62–85 (1946); these Rev. **8**, 282] investigated the probabilities of hits when a rapid-firing gun is used against a moving target. They supposed that a lethal hit was anything in a certain circle with center at a fixed point of the target. The author supposes that the probability (density) of a lethal hit is Gaussian with respect to a coordinate system on the target. The target is k-dimensional. The probability of r hits in an n-round burst is investigated in the general case and under various appropriate simplifying assumptions. Asymptotic theorems are discussed in which $n \to \infty$, and in which n is fixed but certain correlations approach extreme values. Specific applications are given.

J. L. Doob (Ithaca, N. Y.).

Maruyama, Gisirô. The harmonic analysis of stationary stochastic processes. Mem. Fac. Sci. Kyūsyū Univ. A. **4**, 45–106 (1949).

The author gives a survey of the field, including many known results which he apparently obtained independently. Of particular interest are the following results, stated for simplicity in the one-dimensional case, which are apparently new. A stationary Gaussian process with random variables $\{x(t), -\infty < t < \infty\}$ is metrically transitive if its spectral distribution function is continuous. [Reviewer's note: the converse is also true, as can be seen by applying the ergodic theorem to $x(0)^2$.] The process is weakly mixing (that is, the group of shift transformations has this property) if and only if the spectral distribution function is continuous. Dropping the stationarity hypothesis but setting $E\{x(t)\} \equiv 0$, $x(t)$ can be written in the form $x(t) = \int_{-\pi}^{\pi} K(t, s) d\psi(s)$, where the family of random variables $\{\psi(s), -\pi \leqq s \leqq \pi\}$ is a Brownian motion. In all the above results the usual continuity hypothesis is made, that $\lim_{t \to t_0} E\{|x(t) - x(t_0)|^2\} = 0$ for all t_0. If $\{x_n, n \geqq 1\}$ is a stationary Gaussian sequence with $E\{x_1\} = 0$, $E\{x_1^2\} = 1$, if the spectral distribution function F is absolutely continuous, with

$$[2\pi F'(\lambda)]^{\frac{1}{2}} = \alpha + o(\log \log |\lambda|^{-1} / \log |\lambda|^{-1})$$

as $|\lambda| \to 0$, for some $\alpha \geqq 0$, then

$$\limsup_{n \to \infty} \sum_{j=1}^{n} x_j / (2n \log \log n)^{\frac{1}{2}} = \alpha$$

with probability 1. *J. L. Doob* (Ithaca, N. Y.).

***Blanc-Lapierre, A. Considérations sur l'analyse harmonique des fonctions aléatoires.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 61–66. Centre National de la Recherche Scientifique, Paris, 1949.

General review of linear operations on stochastic processes ("filters"), with particular emphasis on $X(t)$ processes given by $X(t) = \int_{-\infty}^{\infty} e^{2\pi i \nu t} dx(\nu)$. Long time averages of $X(t)$ and its transforms and long time correlation averages are evaluated. The meaning for $x(\nu)$ of various types of stationarity of $X(t)$ is investigated. *J. L. Doob* (Ithaca, N. Y.).

***Blanc-Lapierre, A. Analyse harmonique des fonctions aléatoires stationnaires.** Analyse Harmonique, Colloques Internationaux du Centre National de la Recherche Scientifique, no. 15, pp. 121–132. Centre National de la Recherche Scientifique, Paris, 1949. 600 francs.

Expository paper, with particular emphasis on the application of linear operations ("filters") to stationary stochastic processes. *J. L. Doob* (Ithaca, N. Y.).

***Kampé de Fériet, J. Fonctions aléatoires stationnaires et groupes de transformations dans un espace abstrait.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 67–73. Centre National de la Recherche Scientifique, Paris, 1949.

Let $X(\alpha)$ be a random variable for each α. The author points out that the concept of a stationary stochastic process usually considered only for parameter range $\{\alpha\}$ the class of integers or the class of real numbers is most naturally considered for $\{\alpha\}$ any locally compact Abelian group. He shows, for example, how the usual spectral analysis of a stationary process even in this generality is a consequence of known theorems. *J. L. Doob* (Ithaca, N. Y.).

***Wold, H. Sur les processus stationnaires ponctuels.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 75–86. Centre National de la Recherche Scientifique, Paris, 1949.

The author considers a distribution of points on a line. A sample is then a point set, which is supposed to have no finite limit point. Let $\lambda(x, y_1, y_2, \cdots)dt$ be the probability that there will be a point in $(t, t+dt)$ if the preceding points are at $t-x, t-x-y_1, \cdots$. Here $x>0$, $y_j>0$ and the fact that λ does not depend on t makes the distribution stationary in time. In particular, if λ depends only on x, the distribution is a Poisson distribution on the line. The author considers in detail the case when $\lambda=\lambda(x, y_1)$ depends only on x and y_1. [Note by the reviewer. In the Poisson case and more generally in the renewal theory case the distances between successive points are mutually independent random variables. The author's hypothesis that λ may depend on y_1 is equivalent to the hypothesis that the distances between successive points are the random variables of a Markov process.] In particular, when $\lambda=a+by_1^{-1}$ various important distributions connected with the process are calculated explicitly. *J. L. Doob* (Ithaca, N. Y.).

***Ville, J. A. Fonctions aléatoires et transmission de l'information.** Le Calcul des Probabilités et ses Applications. Colloques Internationaux du Centre National de la Recherche Scientifique, no. 13, pp. 115–119. Centre National de la Recherche Scientifique, Paris, 1949.

General discussion. *J. L. Doob* (Ithaca, N. Y.).

Shannon, C. E. Communication theory of secrecy systems. Bell System Tech. J. **28**, 656–715 (1949).

The author adopts a Bayes approach to the analysis of secrecy systems. A secrecy system consists of a collection of keys which transform messages into cryptograms together with a probability distribution of these keys which governs their use. It is supposed that a key is picked in accordance with this distribution, and a message is picked in accordance with a probability distribution depending on the structure of the language. The "enemy," knowing the distributions involved, at least approximately, and intercepting the cryptogram, can calculate the conditional probability of any particular message, in terms of the known cryptogram. He deciphers the cryptogram by maximizing this probability. A calculus of secrecy systems is defined, in terms of various operations, and general classes are described. The effect of enciphering is similar to the effect of noise in communication theory, and the author applies some of the concepts of his earlier work on the latter subject [same J. **27**, 379–423, 623–656 (1948); these Rev. **10**, 133]. For example, quantities are defined which measure the difficulty of obtaining the key used and the message sent, given the cryptogram. These are extremely difficult to calculate in practical cases, but it is shown how to get useful approximations by making the calculations for a certain standard type of secrecy system. *J. L. Doob* (Ithaca, N. Y.).

***Shannon, Claude E., and Weaver, Warren. The Mathematical Theory of Communication.** The University of Illinois Press, Urbana, Ill., 1949. vi+117 pp. $2.50.

The first paper [Shannon] is reprinted from the Bell System Tech. J. **27**, 379–423, 623–656 (1948) [these Rev. **10**, 133]. The second paper [Weaver] . . . consists of an expository introduction to the general theory. . . . In addition, some ideas are suggested for broader application of the fundamental principles of communication theory.

From the preface.

Mathematical Statistics

Seutemann, Karl. Wahrscheinlichkeit und Statistik. Allg. Statist. Arch. **33**, 224–243 (1949).

***Blanche, Ernest Evred. A Systematic Analysis of Frequency Distributions by the Edgeworth Method.** Abstract of a Thesis, University of Illinois, 1941. ii+9 pp.

Richter, H. Zur Gaussischen Verteilung im n-dimensionalen Raume. Z. Angew. Math. Mech. **29**, 161–164 (1949). (German. Russian summary)

The author summarizes well-known properties of linear functions of normally and independently distributed random variables. *E. Lukacs* (Washington, D. C.).

Cohen, A. C., Jr. On estimating the mean and standard deviation of truncated normal distributions. J. Amer. Statist. Assoc. **44**, 518–525 (1949).

Brethouwer, D. H. G. The hyperbolic law of errors. Statistica, Rijswijk **2**, 55–73 (1948). (Dutch. English summary)

Sittig, J. Superposition of two frequency distributions. Statistica, Rijswijk **2**, 206–227 (1948). (Dutch. English summary)

Four orders of moments about the mean of the distribution generated by the superposition of two component distributions, i.e., $dF=pdF_1+qdF_2$, $p+q=1$, are obtained in terms of the corresponding moments of the components. The case of two normal components is considered in some detail. *H. L. Seal* (Toronto, Ont.).

Lindley, D. V. Grouping corrections and maximum likelihood equations. Proc. Cambridge Philos. Soc. **46**, 106–110 (1950).

A correction is provided for use when maximum likelihood estimates derived for ungrouped data are employed with grouped data. *J. L. Hodges, Jr.* (Berkeley, Calif.).

Picard, H. C. The relation between the true distribution, the error distribution and the observable distribution. Statistica, Leiden **3**, 101–107 (1949). (Dutch. English summary)

An observed variate x is supposed to be the sum of an unobservable "true" variate y and a superposed "error"

$x-y$; the distribution of $x-y$ is symmetric about, and dependent upon, y. The rth moment of x about its mean is determined, from the corresponding moment generating function, in terms of functions of the "true" and "superposed" distribution laws, and an indication is given of the corresponding development in the two-dimensional case.

H. L. Seal (Toronto, Ont.).

Radhakrishna Rao, C. A correction to "Note on a problem of Ragnar Frisch." Econometrica **17**, 212 (1949).

The author corrects his previous note [Econometrica **15**, 245–249 (1947); these Rev. **8**, 592], which dealt with the following problem: let $X=a\xi+\alpha$, $Y=b\xi+\beta$, where ξ, α, and β are mutually independent variables, and a and b unknown constants; under what conditions on the distributions of ξ, α, and β will the regressions of X on Y and Y on X be linear? It is noted that the correct solution has been given by Fix [Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946, pp. 79–91, University of California Press, Berkeley and Los Angeles, 1949; these Rev. **10**, 553]. *K. J. Arrow.*

Quenouille, M. H. Note on the elimination of insignificant variates in discriminatory analysis. Ann. Eugenics **14**, 305–308 (1949).

In discriminant function analysis it may be desirable, after computing the discriminant, to omit one or two variates which appear to contribute little. Several methods for doing this without recalculation ab initio are presented. The problem is: given the largest latent root of $(a_{ij}-\lambda b_{ij})$ and its latent vector, to find them when one row and column are deleted from (a_{ij}) and (b_{ij}). *W. G. Cochran.*

Sinha, Gurudas. A note on the expression for the sample estimate of the coefficient of partial correlation. Bull. Calcutta Math. Soc. **41**, 159–161 (1949).

Mitropol'skiĭ, A. K. Ordinary correlation equations. Uspehi Matem. Nauk (N.S.) **4**, no. 5(33), 142–175 (1949). (Russian)

Expository paper on ordinary polynomial regression.

W. Hoeffding (Chapel Hill, N. C.).

Hekker, Th. Construction of chance ellipses in a correlation diagram. Statistica, Leiden **1**, 203–208 (1947). (Dutch. English summary)

van der Laan, E., and Ignatius, J. G. W. Evaluation of a regression by means of orthogonal polynomials. Statistica, Leiden **1**, 219–230 (1947). (Dutch. English summary)

Radhakrishna Rao, C. Representation of 'p' dimensional data in lower dimensions. Sankhyā **9**, 248–251 (1949).

The author states in concise form the known matrix formulation and solution of the problem of the best representation of a set of points in a space of p dimensions on a space of fewer dimensions which is taken to be the equivalent of the minimization of the sum of squares of the distances from the points to the new space. He then shows that the same method can be applied to the representation of k populations of p correlated variables each with identical covariance matrices by the determination of a set of $t(<p)$ vectors in the p-variates such that the scatter of the means of these linear forms over all k populations is a maximum. This is equivalent as he states to the problem of finding the best t linear combinations of the p variates which make the sum of all possible D^2's arising out of a number of populations, calculated with these t variates, a maximum (D^2 is the generalized distance of Mahalanobis).

C. C. Craig (Ann Arbor, Mich.).

Radhakrishna Rao, C. On a transformation useful in multivariate computations. Sankhyā **9**, 251–253 (1949).

Let $x_1, \cdots, x_n$ have a joint distribution with correlation matrix $R=\|r_{ij}\|$. Since computations in multivariate analysis frequently involve R^{-1} it is found convenient to transform to a new set of variables whose correlation matrix is the identity. The transformation is $Y_1=x_1$, $Y_2=x_2-a_{21}Y_1$, $Y_3=x_3-a_{31}Y_1-a_{32}Y_2$, $\cdots$, where the a's are calculated to make the Y's uncorrelated. This method is basically the Gram-Schmidt orthogonalization procedure and the work involved is rather similar to that of the forward solution of the Doolittle inversion method. The advantages of this method seem to be that the work is less than that of a complete inversion and that in many problems after this transformation is made all succeeding operations will involve only simple calculational problems. The author gives applications to discriminant function analysis among other things.

H. Chernoff (Urbana, Ill.).

Robbins, Herbert, and Pitman, E. J. G. Application of the method of mixtures to quadratic forms in normal variates. Ann. Math. Statistics **20**, 552–560 (1949).

The method of mixtures is applied to derive the distribution functions of a positive quadratic form in normal variates and of the ratio of two independent forms of this type. If $X=a(x_m^2+\sum_{i=1}^{r}a_i x_{m_i}^2)$, where the x^2 variates are independent, $a>0$, $a_i\geqq 1$, the distribution of X is determined. Similarly the distribution of Y is found where $Y=X/(x_n^2+\sum_{i=1}^{s}b_i x_{n_i}^2)$, where the variates are independent, $a>0$, $a_i\geqq 1$, $b_i\geqq 1$. The results are extended to cover the corresponding noncentral cases. *L. A. Aroian.*

Chakrabarti, M. C. On the moments of non-central χ^2. Bull. Calcutta Math. Soc. **41**, 208–210 (1949).

Let $X_1, \cdots, X_n$ be n independent stochastic variates having normal distributions with means $\mu_1, \cdots, \mu_n$, respectively, and common variance σ^2; then $\chi'^2=\sum_{i=1}^{n}X_i^2/\sigma^2$ is known as noncentral χ^2. The author finds the central moments, moments about the origin, and the cumulants of χ'^2. *L. A. Aroian* (Culver City, Calif.).

Chand, Uttam. Formulas for the percentage points of the distribution of the arithmetic mean in random samples from certain symmetrical universes. J. Research Nat. Bur. Standards **43**, 79–80 (1949).

The author's summary is as follows. Using the method of Fisher and Cornish, the $100\epsilon\%$ point of the distribution of the arithmetic mean in random samples of size N from any universe having finite cumulants, $K_1, K_2, K_3, \cdots$, is expressed to order $1/N^2$ as a function of (i), the first five positive integral powers of the $100\epsilon\%$ point of a standardized normal variable, and (ii) the quantities K_1, $(K_2)^{\frac{1}{2}}$, and $K_r|(K_2)^r|^2$ for $r=3$, 4, 5, and 6. The numerical coefficients involved are evaluated for the case of sampling from the normal, rectangular, double-exponential, sech, and sech^2 distributions, and the accuracy of the resulting formulas illustrated by numerical examples. *H. Chernoff.*

Hamaker, H. C. A simple technique for producing random sampling numbers. Nederl. Akad. Wetensch., Proc. **52**, 145–150 (1949).

Random sampling numbers were produced by spinning a ten-sided prism. The usual tests of randomness were applied. A fuller description is given in the paper reviewed below.

W. Feller (Ithaca, N. Y.).

Hamaker, H. C. Random sampling numbers. Statistica, Rijswijk **2**, 97–106 (1948). (Dutch. English summary)

Detailed report on the experiments described in the paper reviewed above. *W. Feller* (Ithaca, N. Y.).

Walsh, John E. Concerning compound randomization in the binary system. Ann. Math. Statistics **20**, 580–589 (1949).

Horton's method of adding independent binary digits modulo 2 to reduce their bias [same Ann. **19**, 81–85 (1948); these Rev. **9**, 450], is modified by permitting the same digit to be used more than once. This introduces dependence into the resulting digits, but the author shows an example in which his method wastes fewer digits than does Horton's, while achieving the same control of the maximum absolute difference from $\frac{1}{2}$ of the conditional probability that a digit is 0. *J. L. Hodges, Jr.* (Berkeley, Calif.).

Walsh, John E. On the power function of the "best" t-test solution of the Behrens-Fisher problem. Ann. Math. Statistics **20**, 616–618 (1949).

Scheffé has proposed a test for the equality of means of two normal populations about whose variances nothing is known. The author tabulates the approximate saving in sample sizes which could be made if the variance ratio were known. For example, at the 5 percent level samples of 10 may be reduced to 9. *J. L. Hodges, Jr.* (Berkeley, Calif.).

Matusita, Kameo. Note on the independence of certain statistics. Ann. Inst. Statist. Math., Tokyo **1**, 79–82 (1949).

The author gives a proof of A. T. Craig's theorem [Ann. Math. Statistics **18**, 565–573 (1947); these Rev. **9**, 294] concerning the statistical independence of two bilinear forms in variables drawn obeying a normal multivariate universe.

C. C. Craig (Ann Arbor, Mich.).

Ogawa, Junjiro. On the independence of bilinear and quadratic forms of a random sample from a normal population. Ann. Inst. Statist. Math., Tokyo **1**, 83–108 (1949).

The author gives proofs of the known theorems concerning the statistical independence of quadratic forms drawn (1) from a univariate normal universe and (2) from a multivariate normal universe. Cochran's theorem is also proven.

C. C. Craig (Ann Arbor, Mich.).

Sakamoto, Heihachi. On the criteria of the independence and the degrees of freedom of statistics and their applications to the analysis of variance. Ann. Inst. Statist. Math., Tokyo **1**, 109–122 (1949).

Using the well-known theorems on the independence of linear and quadratic forms in samples from a normal multivariate universe and on their distribution functions which are the subjects of the two preceding papers, the author shows their known applicability to certain well-known test criteria arising in multivariate regression theory and the analysis of variance. *C. C. Craig* (Ann Arbor, Mich.).

Das, A. C. Two-dimensional systematic sampling. Science and Culture **15**, 157–158 (1949).

The author compares the accuracies of random, stratified, and systematic samples from a finite population whose elements are associated with the cells in a two-way layout of strata. Each stratum consists of a two-way layout of cells. The elements of the finite population are assumed to be drawn from an infinite population, and the accuracies are obtained by averaging over all finite populations drawn from the infinite population. *D. F. Votaw, Jr.*

Pompilj, Giuseppe. Sulle medie combinatorie potenziate dei campioni. Rend. Sem. Mat. Univ. Padova **18**, 181–196 (1949).

The author gives a proof of a theorem of C. Gini that the power mean of the combinatorial power means of the same degree in samples drawn without replacements is equal to the like combinatorial power mean of the universe. The theorem is extended in a natural way to samples in which drawings are made with replacements, the proof in both cases employing the factorial moments and allowing the universe supposed finite at first to increase without limit while the elementary probabilities remain constant. The special case for geometric means is also proven without recourse to the limiting process. *C. C. Craig.*

Pompilj, Giuseppe. Sulla significatività delle costanti statistiche. Boll. Un. Mat. Ital. (3) **4**, 112–117 (1949).

This is a brief exposition of the author's views on statistical inference. He is highly critical of the views which prevail in England and the United States.

J. Wolfowitz (New York, N. Y.).

Shone, K. J. Relations between the standard deviation and the distribution of range in non-normal populations. J. Roy. Statist. Soc. Ser. B. **11**, 85–88 (1949).

The author is interested in the estimation of the standard deviation by means of the range for nonnormal populations. Let σ, $\bar{r}$, and σ_r represent the standard deviation, mean range and standard deviation of the range in samples of N, respectively. For samples of 2 for all populations for which these parameters exist $2\sigma^2=\bar{r}^2+\sigma_r^2$. The author shows for $N=3$, for eighteen discrete unimodal populations, by means of numerical analysis, that $\bar{r}/\sigma=2.10-0.81\sigma_r/\bar{r}$ very closely, with similar results for $N=4$ and $N=5$ for five selected populations of extreme form. *L. A. Aroian.*

Goodman, Leo A. On the estimation of the number of classes in a population. Ann. Math. Statistics **20**, 572–579 (1949).

Suppose an urn contains N balls, subdivided into K classes (of varying size and distinguished, e.g., by different colors). The number N is considered as known, K as unknown. A sample of n individuals, drawn without replacement, shows x_1 classes represented by one ball, x_2 classes represented by two balls, etc.; n is supposed to be at least equal to the number of balls in the largest class of the urn. The problem is to estimate K on the basis of the observed numbers $x_1, x_2, \cdots$. It is shown that there exists one and only one unbiased estimate of K. This estimate is a linear combination of the x_i's with explicitly given coefficients. Various modifications of the problem and its solution are discussed. *G. Elfving* (Ithaca, N. Y.).

Madow, William G. On the theory of systematic sampling. II. Ann. Math. Statistics **20**, 333–354 (1949).

It has been noted [W. G. Madow and L. H. Madow, same Ann. **15**, 1–24 (1944); these Rev. **5**, 210] that if an estimate $x=\sum x_i$, the variance of x will be smaller if $\sum_{i\neq j}\{E(x_ix_j)-E(x_i)E(x_j)\}=C<0$ than if x_i, x_j are independent for $i\neq j$, $\sigma^2(x_i)$ being the same in both cases. A sampling design using an x with $C<0$ is said to have negative correlation or, if the distribution of x_i is a chance variable so that C is a chance variable and if $E(C)<0$, expected negative correlation. Under specified conditions, systematic sampling designs have expected negative correlation, so that the estimate has a smaller variance than that from stratified random sampling. The extension to cluster sampling is outlined. Some of the calculations are considerably simplified by use of the facts $E(E_zx)=E_x$ and

$$Exy-ExEy=E\{E_zxy-E_zxE_zy\}+E\{E_zxE_zy-E(E_zx)E(E_zy)\},$$

where E_zx denotes the conditional expectation of x given z.

D. Blackwell (Washington, D. C.).

Maniya, G. M. Generalization of the criterion of A. N. Kolmogorov for an estimate for the law of distribution for empirical data. Doklady Akad. Nauk SSSR (N.S.) **69**, 495–497 (1949). (Russian)

Let $X_1, \cdots, X_n$ be independent identically distributed chance variables with the distribution function $F(x)$. Let $S_n(x)$ be the sample (empirical) distribution function. Define

$$D_n^+(\theta_1,\theta_2)=\sup_{\theta_1\leq F(x)\leq\theta_2}[S_n(x)-F(x)],$$

$$D_n(\theta_1,\theta_2)=\sup_{\theta_1\leq F(x)\leq\theta_2}|S_n(x)-F(x)|.$$

Two theorems are stated without proof. (1) Let $F(x)$ be continuous, $\theta_1^{(n)}=m_1/n=\theta_1+o(n^{-\frac{1}{2}})$, and $\theta_2^{(n)}=m_2/n=\theta_2+o(n^{-\frac{1}{2}})$, $0<\theta_1<\theta_2<1$. Then

$$\lim_{n\to\infty}P\{D_n^+(\theta_1^{(n)},\theta_2^{(n)})\leqq\lambda/\sqrt{n}\}$$
$$=\frac{1}{2\pi(1-R^2)^{\frac{1}{2}}}\int_{-\infty}^{a}\int_{-\infty}^{b}\exp\left[-\tfrac{1}{2}\theta(z_1,z_2)\right]dz_1dz_2$$
$$-\frac{e^{-2\lambda^2}}{2\pi(1-R^2)^{\frac{1}{2}}}\int_{-\infty}^{a'}\int_{-\infty}^{b'}\exp\left[-\tfrac{1}{2}\theta(z_1,z_2)\right]dz_1dz_2,$$

$$a=\frac{\lambda}{[\theta_1(1-\theta_1)]^{\frac{1}{2}}},\quad b=\frac{\lambda}{[\theta_2(1-\theta_2)]^{\frac{1}{2}}},\quad a'=\frac{\lambda-2\lambda\theta_1}{[\theta_1(1-\theta_1)]^{\frac{1}{2}}},$$

$$b'=\frac{\lambda-2\lambda(1-\theta_2)}{[\theta_2(1-\theta_2)]^{\frac{1}{2}}},\quad R=[\theta_1(1-\theta_2)]^{\frac{1}{2}}/[\theta_2(1-\theta_1)]^{\frac{1}{2}},$$

$$\theta(z_1,z_2)=(1-R^2)^{-1}[z_1^2+2Rz_1z_2+z_2^2],$$

$$\tilde{\theta}(z_1,z_2)=(1-R^2)^{-1}[z_1^2-2Rz_1z_2+z_2^2].$$

(2) Under the conditions of theorem 1,

$$\lim_{n\to\infty}P\{D_n(\theta_1^{(n)},\theta_2^{(n)})\leqq\lambda/\sqrt{n}\}$$
$$=\frac{1}{2\pi(1-R^2)^{\frac{1}{2}}}\int_{-a}^{a}\int_{-b}^{b}\exp\left[-\tfrac{1}{2}\theta(z_1,z_2)\right]dz_1dz_2$$
$$-\pi^{-1}(1-R^2)^{-\frac{1}{2}}\sum_{k=1}^{\infty}(-1)^{k-1}e^{-2k^2\lambda^2}$$
$$\times\int_{-\alpha_k}^{\alpha_k}\int_{-\beta_k}^{\beta_k}\exp\left[-\tfrac{1}{2}\theta(z_1,z_2)\right]dz_1dz_2,$$

$$\alpha_k=(\lambda-2k\lambda\theta_1)/[\theta_1(1-\theta_1)]^{\frac{1}{2}},$$

$$\beta_k=[\lambda-2k\lambda(1-\theta_2)]/[\theta_2(1-\theta_2)]^{\frac{1}{2}}.$$

It is possible that a misprint has occurred and that one of the θ's in (2) should be $\tilde{\theta}$. The reviewer has not checked this, but it could be checked by specialization and comparison with Kolmogoroff's known result. *J. Wolfowitz.*

Smirnov, N. V. On the Cramér-Mises criterion. Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 196–197 (1949). (Russian)

The author sketches another derivation of the asymptotic limiting distribution of the ω^2 criterion measuring the discrepancy between a given distribution and the empirical one obtained from a sample. [Cf. his earlier paper, C. R. Acad. Sci. Paris **202**, 449–452 (1936).] *J. L. Doob.*

Wald, Abraham. Note on the consistency of the maximum likelihood estimate. Ann. Math. Statistics **20**, 595–601 (1949).

The author gives a simple proof, based on the strong law of large numbers, of the consistency of the maximum likelihood estimate under certain conditions. *D. Blackwell.*

Wolfowitz, J. On Wald's proof of the consistency of the maximum likelihood estimate. Ann. Math. Statistics **20**, 601–602 (1949).

A variation of Wald's proof of consistency of the maximum likelihood estimate [see the preceding review]. It uses only the weak law of large numbers and consequently applies to a wider class of variables. *D. Blackwell.*

Wolfowitz, J. The power of the classical tests associated with the normal distribution. Ann. Math. Statistics **20**, 540–551 (1949).

A simple unified method is given for proving the optimum properties (first obtained by Hsu, Simaika and Wald) of certain classical tests associated with univariate and multivariate normal distributions. The author uses the invariance of the statistical problems under suitable groups of linear transformations. *E. L. Lehmann* (Berkeley, Calif.).

Sobel, Milton, and Wald, Abraham. A sequential decision procedure for choosing one of three hypotheses concerning the unknown mean of a normal distribution. Ann. Math. Statistics **20**, 502–522 (1949).

A sequential decision procedure is formulated for the following multi-decision problem: to assign the unknown mean θ of a normally distributed random variable with known variance to one of three given intervals $(-\infty, a_1)$, (a_1, a_2), (a_2, ∞). It is also required that the probability of a wrong decision does not exceed prescribed upper bounds. In essence the procedure requires observations to be taken until decisions are made, using Wald's sequential probability ratio test, with respect to both hypotheses (H_1) $\theta\leqq a_1$ against $\theta>a_1$; (H_2) $\theta\leqq a_2$ against $\theta>a_2$. The operating characteristic and close bounds on the expected number of observations are determined. The authors point out that the procedure is not optimum but appears to be near-optimum and has the advantage of simplicity. It is shown to be more efficient than a nonsequential procedure.

D. G. Chapman (Seattle, Wash.).

Arrow, K. J., Blackwell, D., and Girshick, M. A. Bayes and minimax solutions of sequential decision problems. Econometrica **17**, 213–244 (1949).

As extensively indicated in its initial footnote, this paper intimately overlaps recent published and unpublished work by Wald and Wolfowitz on the theory of sequential analysis.

[Cf. Wald, Econometrica 15, 279–313 (1947); Wald and Wolfowitz, Ann. Math. Statistics 19, 326–339 (1948); Proc. Nat. Acad. Sci. U. S. A. 35, 99–102 (1949); these Rev. 9, 454; 10, 201, 466.] For the most part the theorems proved here are extensions of theorems due to Wald and Wolfowitz. The proof of one theorem seems to be more rigorous than the corresponding proof of Wald and Wolfowitz, in that the measurability of a certain set which Wald and Wolfowitz seem to have assumed measurable is proved. Some examples of dichotomous and trichotomous sequential decision problems of academic interest are given in detail.

L. J. Savage (Chicago, Ill.).

Cox, D. R. The use of the range in sequential analysis. J. Roy. Statist. Soc. Ser. B. 11, 101–114 (1949).

The author constructs sequential tests, based on the range of the sample, for the following two hypotheses: (1) $\sigma=\sigma_0$ against the alternative $\sigma=\sigma_1$, σ being the standard deviation of a normal distribution, (2) $\sigma_2=\sigma_1$ against the alternative $\sigma_2=\phi\sigma_1$, σ_1 and σ_2 being the unknown standard deviations of two normal distributions and ϕ being a specified constant. The distribution of the range of a sample from a normal distribution is approximated by the distribution $F(a, b)$ with density function $(x/a)^{b-1}\varepsilon^{-x/a}/a\Gamma(b)$, $x>0$. The closeness of the approximation is discussed. With this approximation, (1) and (2) become the following hypotheses about the parameters a, b in $F(a, b)$: (1′) $a=a_0$ against $a=a_1$, b known; and (2′) $a_2=a_1$ against $a_2=\phi a_1$, a_1, a_2 unknown, b known and the same for both distributions. The optimum test for (1′) is the sequential probability ratio test. The author presents a test of (2′) for which α, β are independent of a_1; it is obtained as the limit as $r\to\infty$ of the optimum test when a_1 has a rectangular distribution over $(0, r)$.

D. Blackwell (Washington, D. C.).

Quenouille, M. H. Problems in plane sampling. Ann. Math. Statistics 20, 355–375 (1949).

There are numerous ways in which a rectangular area can be sampled, according as the sampling in a single dimension is random (r), stratified (st), or systematic (sy). The variance of the mean is given for nine combinations that might be used in practice. If the linear correlograms are monotone-decreasing functions of the distance, the effect of alignment (i.e., of placing the sampling-units in lines parallel to the sides) is to increase the variance for r and st, but may be in either direction for sy. Limiting forms of the variances are given in large samples. From these, the relative efficiencies of different types of sample are discussed in the cases $\rho_{uv}=\rho_u\rho_v$ and $\rho_{uv}=\varepsilon^{-r}$, where ρ_{uv} is the correlation between two units whose distance apart is $r=(u^2+v^2)^{\frac{1}{2}}$. The author speculates that the best type of systematic sampling will be superior to stratified random sampling in a wide variety of cases. Three methods for estimating the variance from a sample are compared. An adjustment is given to handle linear trends in the area. The paper also gives an extension of Cochran's results [same Ann. 17, 164–177 (1946); these Rev. 8, 43] for one-dimensional sampling.

W. G. Cochran (Raleigh, N. C.).

Quenouille, M. H. Approximate tests of correlation in time-series. J. Roy. Statist. Soc. Ser. B. 11, 68–84 (1949).

The author considers the partial correlation coefficient $r_{12\cdot 23\cdots(k-1)}$ of a time-series as a possible test quantity in the construction of autoregressive schemes. Its suitability is that, if the series obeys an autoregressive scheme of order p, then $r_{12\cdot 23\cdots(k-1)}$ has expectation zero for $n>p+1$, in large samples at least. Thus, by previous work, it may be tested in the same way as an ordinary correlation coefficient of uncorrelated pairs of variates, if 3 extra degrees of freedom are allowed. The author now investigates numerically the bias in small samples, and the extent to which it is affected by different definitions of the coefficients. The principal bias is seen to be of order n^{-1}, and a method is suggested for eliminating it. The application of the partial-correlation test to cross-correlations between time-series is also discussed.

P. Whittle (Uppsala).

*Mann, H. B. **Analysis and Design of Experiments. Analysis of Variance and Analysis of Variance Designs.** Dover Publications, Inc., New York, N. Y., 1949. x+195 pp. $2.95.

This little volume brings together most of the mathematical material associated with the analysis of variance and the construction of experimental designs. The following list of abbreviated chapter headings reveals the content: chi-square and F distributions, multivariate normal distribution, analysis of a one-way classification, linear hypotheses, r-way classification, power of analysis of variance tests, analysis of latin squares and incomplete block designs, Galois fields and orthogonal latin squares, construction of incomplete block designs, non-orthogonal data, factorial experiments, randomized blocks and quasifactorial designs, analysis of covariance, interblock estimates and variances. Since all this material is covered in 177 pages the presentation is necessarily much condensed and in fact only the bare mathematics is given in most chapters. There is practically no motivation nor explanation and there is hardly a word on the art of designing experiments. The general components-of-variance model is not discussed at all. Also the book was hurriedly written and carelessly proof-read so that it contains many misprints, minor errors, and confusing notations. The overall result is a book that is difficult to read even for a person not new to the subject.

Nevertheless, this book will be a welcome addition to the library of the mathematical statistician for it provides the first connected account of an area which is rapidly becoming quite important to statistics and to scientific experimentation in general. The best parts of the book are those devoted to the construction of designs, where the author uses the methods of the Indian workers. This part of the book is complete in that the necessary ideas from abstract algebra are developed from scratch and are well illustrated by simple examples. Most attention is given to the construction of orthogonal latin squares and balanced incomplete block designs.

The author has asked the reviewer to mention that W. L. Stevens should have been given credit for the methods of constructing orthogonal latin squares, as well as R. C. Bose, who is credited in the book. They did the work independently at about the same time.

A. M. Mood.

Banerjee, K. S. On certain aspects of spring balance designs. Sankhyā 9, 367–376 (1949).

This studies the efficiency of certain designs for the weighing problem which the reviewer suggested [Ann. Math. Statistics 17, 432–446 (1946); these Rev. 8, 478] for cases in which maximum efficiency cannot be achieved. The connection between weighing designs and balanced incomplete block designs is discussed with particular reference to variance factors. Also $\frac{3}{4}$ replicates of 2^n factorial designs are applied to the weighing problem.

A. M. Mood.

Stoffels, A. The treatment of field experiments by Knut Vik's method. Statistica, Leiden 1, 209–218 (1947). (Dutch. English summary)

Pages 216–218 contain an editorial postscript. Vik's method was published in Nordisk Jordbruksforskning 1923–24, 429–436; Meldinger fra Norges Landbrukshøiskole 1924, 129–181.

Stoffels, A. The treatment of the results of field experiments by reducing the yields in two directions. Statistica, Rijswijk 2, 242–249 (1949). (Dutch. English summary)

NUMERICAL AND GRAPHICAL METHODS

***Berkeley, Edmund Callis. Giant Brains or Machines That Think.** John Wiley & Sons, Inc., New York, N. Y., 1949. xvi+270 pp. $4.00.

"This book is intended for everyone." Chapters are devoted to punched card equipment, the Harvard-IBM automatic sequenced controlled computer, the ENIAC, the Bell Relay Calculator, and the Kahn-Burkhart logical truth calculator. There is an extensive bibliography. In addition, there are discussions, for instance, of "the nature of language and of symbols, the meaning of thinking, the human brain and nervous system, the future design of machinery that can think, and a little algebra and logic." *J. Todd.*

Zuse, Konrad. Über den Allgemeinen Plankalkül als Mittel zur Formulierung schematisch-kombinativer Aufgaben. Arch. Math. 1, 441–449 (1949).

The author discusses some aspects of a scheme for a machine to solve combinatorial problems. The body of the paper is devoted to a discussion of a logical calculus which he calls "allgemeine Plankalkül." It proceeds by illustrative examples since he says he does not have space for a more formal discussion. His over-all scheme is somewhat suggestive of Turing's thesis. There are no suggestions given as to whether the author believes that his machine can be realized as a physical device. *H. H. Goldstine.*

Andrews, E. G., and Bode, H. W. Use of the relay digital computer. Elec. Engrg. 69, 158–163 (1950).

van Wijngaarden, A. General survey of modern computing machines. Nederl. Tijdschr. Natuurkunde 15, 244–254 (1949). (Dutch)

Manneback, Ch. Sur les grandes machines mathématiques américaines. Bull. Soc. Math. Belgique 1 (1947–1948), 15–16 (1949).

Ekelöf, Stig. Macchine calcolatrici automatiche. Elettrotecnica 36, 306–317 (1949).

Jowett, G. H. The calculation of sums of squares and products on a desk calculating machine. J. Roy. Statist. Soc. Ser. B. 11, 89–90 (1949).

Oppelt, Winfried. Graphische Verfahren zur komplexen Multiplikation. Arch. Elektr. Übertragung 2, 76–78 (1948).

***Emde, Fritz. Tafeln Elementarer Funktionen. Tables of Elementary Functions.** 2d ed. B. G. Teubner Verlagsgesellschaft, Leipzig, 1948. xii+181 pp. 3.50 DM.

This is an almost unchanged reprint of the edition of 1941; cf. these Rev. 3, 152.

***Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments.** Prepared by U. S. Department of Commerce, National Bureau of Standards, The Computation Laboratory. Mathematical Table MT3, 2d ed. United States Government Printing Office, Washington, D. C., 1949. xviii+405 pp. $2.50.

The first edition was published in New York in 1939; cf. these Rev. 2, 64. Three misprints have been corrected and the accuracy of the conversion table has been increased.

Salzer, Herbert E., and Zucker, Ruth. Table of the zeros and weight factors of the first fifteen Laguerre polynomials. Bull. Amer. Math. Soc. 55, 1004–1012 (1949).

The zeros $x_i^{(n)}$ are given to 12 decimals and the weight factors $\alpha_i^{(n)}$ to 12 significant figures. In addition, following A. Reisz [Ark. Mat. Astr. Fys. 29A, no. 29 (1943); these Rev. 6, 218], the product $\omega_i^{(n)}$ of each weight factor by the exponential of the corresponding zero is given to 12 significant figures; these products facilitate approximate quadratures of the form

$$\int_0^\infty F(x)dx=\int_0^\infty e^{-x}(e^xF(x))dx\sim\sum\alpha_i^{(n)}(e^{x_i^{(n)}}F(x_i^{(n)}))$$
$$=\sum\omega_i^{(n)}F(x_i^{(n)}).$$

The method of computation and checking is described and the results are compared with earlier tables. *J. Todd.*

Hall, R. N. The application of non-integral Legendre functions to potential problems. J. Appl. Phys. 20, 925–931 (1949).

Legendre functions of nonintegral degree occur in potential problems, when ν is often determined by the condition that for a given μ_0 $(-1<\mu_0<1)$ the equation $P_\nu(\mu_0)=0$ is satisfied. The roots of this equation are ν_s and $-\nu_s-1$ $(s=0, 1, 2, \cdots)$. The author gives for $s\leqq 0, 1, \cdots, 5$ and $-1<\mu_0<1$ graphs of ν_s and more accurate diagrams of $(\nu_s-s)^{-1}$. He also gives graphs of $H_s=\int_{\mu_0}^1[P_{\nu_s}(\mu)]^2d\mu$ and of $H_s^{-1}\int_{\mu_0}^1 P_{\nu_s}(\mu)d\mu$. The paper contains some supplementary material, and an indication of the application of the results to potential problems. *A. Erdélyi* (Pasadena, Calif.).

Stewart, J. C. Numerical computation of the elliptic function sn $\left(\frac{mK}{n}\right)$, where m and n are integers. Communication Rev. 1, 24–29 (1948).

In the design of certain electric wave filters, values of the Jacobian elliptic function sn u are required for $u=mK/n$, where n and m are integers, and $4K$ is the real period of sn u. It is stated that available tables would provide only five significant figures whereas the author requires eight. To overcome this difficulty he suggests a new procedure of numerical calculation. Two theorems are used: (1) the duplication formula for sn u, (2) the relation existing between sn u and sn v when $u\pm v$ is an odd multiple of K. The case $n=2^r$ $(r=0, 1, 2, \cdots)$ is almost trivial. For n odd, let d be the smallest positive solution of $2^d\equiv 1 \pmod n$. Then

$d+1$ cyclic relations between sn $(2^sK/n)$ $(s=0, 1, 2, \cdots, d)$ are obtained, which can be solved to any number of decimals if the underlying iterative process is started from an approximate value of any one of these $d+1$ values. The remaining classes of fractions in the set m/n can be treated analogously. Finally, for even values of n the process is reducible to that for n odd, in combination with theorem (1). [Reviewer's remark. The accuracy required by the author for the values of sn (mK/n) is now easily provided by interpolation in the recent twelve-decimal tables of G. W. and R. M. Spenceley, Smithsonian Elliptic Functions Tables, Smithsonian Institution, Washington, D. C., 1947; these Rev. **9**, 380].

C. J. Bouwkamp (Eindhoven).

Petrović, Mihail. Approximate representations of elliptic functions by means of elementary functions. Glas Srpske Akad. Nauka **189**, 49–70 (1946). (Serbian)

It is shown that for real values of x we have, with a surprisingly good approximation, sn $x \cong \sin \pi x/k$, where $2k$ is the real period of sn x (that is, sn $2k=1$). Similar approximations hold for other elliptic functions. The paper is of a computational nature. *W. Feller* (Ithaca, N. Y.).

Chandrasekhar, S., and Wares, Gordon W. The isothermal function. Astrophys. J. **109**, 551–554 (1949).

The differential equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\psi}{d\xi}\right)=e^{-\psi}$$

governs the distribution of matter in hydrostatic equilibrium under its own gravitation and at constant temperature. The paper gives tables connected with the solution for which $\psi=d\psi/d\xi=0$ at $\xi=0$. For $\xi=0(0.1)5.2(0.2)10(1)30(2)54$, $55(5)180$, $200(25)750(50)2200$, values are given of ψ and $d\psi/d\xi$ to 7 or 8 decimals, of $e^{-\psi}$ to 7 decimals, or 7 or 8 figures, and of $u=(\xi e^{-\psi})/(d\psi/d\xi)$ and $v=\xi d\psi/d\xi$ to 6 decimals. For ξ up to 10, the tables give also $\xi^2 d\psi/d\xi$ and $(\frac{1}{3}\xi)/(d\psi/d\xi)$ to 5 or 6 decimals. *J. C. P. Miller.*

Chandrasekhar, S. The functions $G_{n,m}(\tau)$ and $G'_{n,m}(\tau)$ of order 6 ($m=6$ and $m \geqq n$). Astrophys. J. **109**, 555–556 (1949).

The paper contains a table supplementing one in an earlier paper of the author [same J. **108**, 92–111 (1948); these Rev. **10**, 543], and a correction to the headings of the original table. *F. Smithies* (Cambridge, England).

Frame, J. S. An approximation to the quotient of gamma functions. Amer. Math. Monthly **56**, 529–535 (1949).

The author gives a table for

$$F_n(u)=\log \Gamma(n+\tfrac{1}{2}+\tfrac{1}{2}u)-\log \Gamma(n+\tfrac{1}{2}-\tfrac{1}{2}u) -\tfrac{1}{2}u \log (n^2+\tfrac{1}{12}+\tfrac{1}{12}u^2)$$

$(n=1, 2, 3, 4, \infty$ and $u=0, \frac{1}{2}, 1, 2)$. For $n \geqq 1$, $-1 \leqq u \leqq 1$ we have $0<F_n(u)<1$ and for $n \geqq 1$, $-2 \leqq u \leqq 2$ a good estimate for $F_n(u)$ is given by

$$(n^2+0.33+0.06(u^2-1)n^{-2})(n^2+1-0.20u^2)^{-1}.$$

The author moreover deduces the asymptotic expansion for $F_n(u)$ for large n. *J. G. van der Corput* (Amsterdam).

Mogno, R. Su una formula approssimata per il calcolo di $n!$ e le sue applicazioni. Metron **14**, 67–77 (1940).

In an obscure and completely unrigorous way the author arrives, from a combination of the formulae of Stirling and Wallis, at the correct result $n! \simeq n^n e^{-n}(2\pi n)^{\frac{1}{2}}(1+\frac{1}{2}n^{-1})^{\frac{1}{2}}$, which he recommends as giving a closer approximation than Stirling's formula without being more laborious. Although this cannot be denied, the known simple expression $n! \simeq n^n e^{-n+1/12}(2\pi n)^{\frac{1}{2}}$ seems to be still better.

S. C. van Veen (Delft).

Jecklin, H., und Zimmermann, H. Eine praktische Interpolationsformel. Mitt. Verein. Schweiz. Versich.-Math. **48**, 126–144 (1948).

A procedure is given for interpolating from three given values by fitting to them a quotient of two linear functions. This is equivalent to applying Thiele's interpolation formula [L. M. Thomson, The Calculus of Finite Differences, Macmillan, London, 1934, chapter 5] as far as second order reciprocal differences. *T. N. E. Greville.*

Salzer, Herbert E. Formulas for complex Cartesian interpolation of higher degree. J. Math. Physics **28**, 200–203 (1949).

W. J. Taylor's scheme for real Lagrangian interpolation is generalised by the author for complex Lagrangian interpolation. Let $f(z)$ be an analytic function of the complex variable z, tabulated in a square grid of side lengths 1. Then $f(z)=f(z_0+P)$ can be interpolated by a polynomial of degree m using the points z_0+k and the values $f_k=f(z_0+k)$, where k runs through $m+1$ complex values, so that $f(z)=\sum_k a_k f_k/\sum_k a_k$, where $a_k=A_k/(P-k)$, $A_k=1/\prod'_{j,k}(k-j)$. The author gives the values of A_k for $m=2, 3, \cdots, 8$. For $m=4, \cdots, 8$ several sets of the $m+1$ points z_0+k are given with the values of the A_k.

E. Bodewig (The Hague).

Baum, Richard F. A contribution to the approximation problem. Proc. I.R.E. **36**, 863–869 (1948).

One of the problems in network synthesis is to approximate a given curve by a rational fraction of the form $|Z(j\omega)|^2(A_0+A_2\omega^2+A_4\omega^4+\cdots)/(B_0+B_2\omega^2+B_4\omega^4+\cdots)$. The author reviews some methods used for the solution of this problem and then proposes the use of "semi-infinite slopes." These are attenuation functions which when plotted on a logarithmic frequency scale are constant for frequencies below a given ω_0 and rise or fall at constant rate beyond ω_0. The desired curve in logarithmic form is approximated by a sum of such functions and these in turn are represented by Butterworth functions $B_{2n}(\omega^2)=1+\omega^{2n}$, so that the desired rational fraction has the form

$$|Z(j\omega)|^2=\frac{[1+(\omega/\omega_1)^2]^a[1+(\omega/\omega_2)^2]^b\cdots}{[1+(\omega/\omega_3)^2]^c[1+(\omega/\omega_4)^2]^d\cdots}.$$

An advantage of this form is the ease with which the zeros and poles can be obtained. *W. E. Milne.*

Harrison, Joseph O., Jr. Piecewise polynomial approximation for large-scale digital calculators. Math. Tables and Other Aids to Computation **3**, 400–407 (1949).

"Given the function $f(x)$ defined on the interval $[\alpha, \beta]$, a specified positive tolerance T, and a specified positive integer N," it is "required to divide $[\alpha, \beta]$ into sub-intervals $[c_{i-1}, c_i]\cdots$ and to determine nth degree polynomials $P_n^i(x)$ with $n \leqq N$, such that the upper bound of $|f(x)-P_n^i(x)| \leqq T$ on $[c_{i-1}, c_i]$." The discussion is restricted to approximation by nth degree polynomials agreeing with $f(x)$ at $n+1$ points on $[c_{i-1}, c_i]$. Assuming the $n+1$ points of agreement to have

the same relative positions on each subinterval, and these relative positions to be given, the author develops a procedure for determining the subintervals so as to satisfy the required conditions. He shows that, under the procedure given, the number of subintervals is a minimum when the relative positions of the points of agreement on each subinterval are the same as the relative positions on the interval $[-1, 1]$ of the zeros of the Chebyshev polynomial "of the first kind" of order $n+1$ defined by $T_{n+1}(u)=2^{-n}\cos[(n+1)\cos^{-1}u]$. *T. N. E. Greville.*

van der Poel, W. L. Approximation of experimentally given functions by means of polynomials with Tchebycheff's method. Simon Stevin **26**, 189–199 (1949). (Dutch)

Given the tabular function $f(x_k)=y_k$ $(k=1, \cdots, n)$, the problem of Tchebycheff is to approximate the function $f(x)$ by a polynomial $\sum_{l=0}^{m}a_{ml}x^l$ of least possible degree m in such a way that the sum of the squares of the errors

$$\epsilon_m=\sum_{k=1}^{n}\left(\sum_{l=0}^{m}a_{ml}x_k^l-y_k\right)^2$$

will be a minimum. The author defines the functions $\psi_m(x)=\sum_{l=0}^{m}\delta_{ml}x^l$, where δ_{ml} is the cofactor of b_{ml} in the determinant $\Delta_m=|b_{lr}|$ of order m; $b_{lr}=S_{l+r}=\sum_{k=1}^{n}x_k^{l+r}$; $\delta_{m,m}=\Delta_{m-1}$. The author states that the functions $\psi_m(x)$ are orthogonal:

$$\sum_{k=1}^{n}\psi_l(x_k)\cdot\psi_s(x_k)=0, \qquad l\neq s,$$

$$=\delta_{ss}\Delta_s=\Delta_{s-1}\Delta_s, \qquad l=s.$$

Consequently $\Delta_s>0$ $(s=0, 1, \cdots)$. If $U_m=|c_{lr}|/\Delta_{m-1}$; $c_{lr}=S_{l+r}$ $(r<m)$, $c_{lr}=\sum_{k=1}^{n}y_kx_k^l$ $(r=m)$, the best approximated polynomial of order m is $\sum_{l=0}^{m}a_{ml}x^l=\sum_{l=0}^{m}(U_l/\Delta_l)\psi_l(x)$. The sum of the squares of the errors $\epsilon_m=\epsilon_{m-1}-U_m^2/\Delta_m$ consequently decreases with m. The formulae are simplified for the case of equidistant abscissae. In that case the functions $\psi_m(x)$ are closely related to the Legendre polynomials $P_m(2x-1)$. *S. C. van Veen* (Delft).

Simaika, J. B. On a method of curve fitting and its application to the variation of the note issue in Egypt. Proc. Math. Phys. Soc. Egypt **3**, 65–72 (1948).

The author's summary states: "The problem of fitting a curve to a function which can be represented by the sum of a polynomial and periodic terms is treated here by the use of mixed orthogonal polynomials: a combination of Legendre polynomials, sines and cosines, when the variable is continuous or a combination of Tschebyscheff polynomials, sines and cosines when the variable is discrete. Moreover, an explicit expression for the mixed polynomials is given, in both cases, when the ordinary polynomial is of the second degree." *T. N. E. Greville.*

Jordan, Charles. Note on approximation and graduation by orthogonal moments. Hungarica Acta Math. **1**, no. 4, 4–9 (1949).

This is a summary of results obtained in the author's previous publications on the subject [e.g., Proc. London Math. Soc. (2) **20**, 297–325 (1922); J. Soc. Hongroise Statistique [Magyar Statisztikai Tarsasag] **7**, 567–599 (1929); Ann. Math. Statistics **3**, 257–357 (1932); Calculus of Finite Differences, Budapest, 1939; these Rev. **1**, 74].

T. N. E. Greville (Washington, D. C.).

Heinrich, G. Auswertung von Stossmessungen mittels Laplace-Transformation. Österreich. Ing.-Arch. **3**, 327–336 (1949).

If the temporal course of an acceleration is measured with an accelerometer of any type, the acceleration curve will generally not be quite correct. In case of temporally limited phenomena, a rectification of the curve is possible by using the damping curve that is obtained from the apparatus after a very short inelastic shock of fixed intensity. The influence of the latter is accounted for by an integral equation of the Volterra type, which can be solved by using the Laplace transformation. A graphical method is thereby developed which makes rectification possible.

A. E. Heins (Pittsburgh, Pa.).

Stange, K. Die zweckmässige Auswertung von punktweise aufgenommenen Zeit-Weglinien. Ing.-Arch. **16**, 383–402 (1948).

It is supposed that a sequence of points at equal intervals on a curve have been determined with a given precision and that a smoothed value of the ordinate to the curve, as well as values of the first and second derivatives, are obtained at each point by fitting a polynomial by the method of least squares to a specified number of neighboring points. The extent of reduction of error in the ordinate and the derivatives effected through the smoothing and the amount of distortion resulting therefrom are discussed in relation to the degree of polynomial and the number of neighboring points used in the smoothing process. *T. N. E. Greville.*

Stange, K. Das Bildungsgesetz für die Fehlerformeln beim Ausgleichen von fehlerhaften Messreihem mit Hilfe ganzer rationaler Funktionen wachsender Ordnung. Z. Angew. Math. Mech. **29**, 225–238 (1949). (German. Russian summary)

A set of measured quantities $\cdots, x_{k-1}, x_k, x_{k+1}, \cdots$ corresponding to values of an argument at equal intervals of length τ can be smoothed by a formula (1) $\bar{x}_k=\sum_{\nu=-n}^{n}\lambda_\nu x_{k+\nu}$, where the λ's are determined from the best fitting (in the sense of least squares) polynomial of degree $R=2r$, or $R=2r+1$. The author investigates the asymptotic behavior, as the n in (1) grows large, of what he calls the "physical error," namely $f_P=(\sum_{\nu=-n}^{n}\lambda_\nu^2\mu)^{\frac{1}{2}}$, and of what he calls the "mathematical error" f_M, where

$$f_M=\frac{x_k^{(2r+2)}\tau^{2r+2}}{(2r+2)!}\sum_{\nu=-n}^{n}\nu^{2r+2}\lambda_\nu.$$

To do this he expressed f_P and f_M by means of ratios of certain determinants. First and second order approximations are worked out and the accuracy of the second approximation is indicated by graphical comparison of the approximate with the true values. He also obtains formulas for the $\bar{x}$ expressed as functions of differences of even order and treats in considerable detail the special case $n=r+1$, i.e., the case where two more points are used than there are coefficients in the polynomial. *W. E. Milne.*

Brandler, František. Contribution à la résolution numérique de l'équation du troisième degré. Časopis Pěst. Mat. Fys. **74**, D54–D63 (1949). (Czech. French summary)

En employant la méthode des moindres carrés, l'auteur résout les équations algébriques du troisième degré.

Author's summary.

Sponder, Erich. Ein Näherungsverfahren zur Lösung algebraischer Gleichungen 4. Grades. Elemente der Math. **4**, 116–117 (1949).

The method described is identical with method A of Porter and Mack [Philos. Mag. (7) **40**, 578–585 (1949); these Rev. **11**, 57]. *E. Bodewig* (The Hague).

Bilimovitch, Anton D. On a geometrical construction and apparatus for approximate solution of Kepler's equation. Glas Srpske Akad. Nauka **191**, 117–124 (1948). (Serbian. English summary)

A simple geometrical construction is given which yields the solution of the equation $\bar{u}=w+\sin^{-1}(e\sin\bar{u})$. It is equivalent to an alignment chart with uniform scales for w and $\bar{u}$ on a circle, a uniform scale for e on a diameter, a fixed point at the center and parallel indices. When the series for $\bar{u}$ in powers of e is compared with the similar series for u as given by Laplace [Oeuvres Complètes, v. 1, p. 198], where $u=w+e\sin u$, it is found that

$$\bar{u}-u=\tfrac{1}{2}e^3\sin^3 w+\tfrac{1}{2}e^4\sin^3 w\cos w+\cdots.$$

[The author determined only the first term of this error; he obtained the coefficient $-\frac{1}{3}$ instead of $\frac{1}{2}$.] A mechanical device materializing the geometrical relations is described. Simplicity is thought to justify this addition to the long list of graphical solutions of Kepler's equation.

R. Church (Annapolis, Md.).

Boothroyd, A. R., and Cherry, E. Colin. Analogue calculating machine for functions of a complex variable. Nature **163**, 687 (1949).

Rational functions of a complex variable are studied by analogue methods. The analogue used is a current-carrying conducting sheet on which are placed point sources of current to represent singularities. Exploring electrodes can be used to measure either the potential or the potential gradient at selected points or along various paths on the conducting sheet. These measurements furnish data which may be interpreted in terms of properties of the function represented. *S. H. Caldwell* (Cambridge, Mass.).

Weber, R. Sur les méthodes de calcul employées pour la recherche des valeurs et vecteurs propres d'une matrice. Recherche Aéronautique **1949**, no. 10, 57–60 (1949).

Verfasser gibt eine einfache, nie versagende Eliminationsmethode zur genauen Bestimmung der charakteristischen Gleichung $\det T=\det(A-tB)=0$. Sie kommt darauf hinaus, dass T auf Dreiecksform D gebracht wird, so dass also $\det T=\det D$ (bis auf einen Zahlenfaktor). Nach Bestimmung von t ergibt sich aus den einzelnen Zeilen von D der zu t gehörende Eigenvektor. Als Rezensent habe ich den Eindruck, dass Verfasser die Methode überschätzt. In der Tat erfordert sie zur Bestimmung von $\det(A-tE)$ mehr Operationen als die Methoden von Danielewski und Samuelson und fast ebenso viele wie die (allerdings ungenaue) Methode von Kryloff-Lusin-Duncan [Krylov, Bull. Acad. Sci. URSS (7) Cl. Sci. Math. Nat. [Izvestiya Akad. Nauk SSSR] **1931**, 491–539; Luzin, ibid., 903–958; Frazer, Duncan, and Collar, Elementary Matrices and Some Applications to Dynamics and Differential Equations, Cambridge University Press, Macmillan, New York, 1946; diese Rev. **8**, 365]. Doch ist die Methode zweifellos eine wesentliche Bereicherung bestehender Methoden.

Eine andere Methode des Verf., um umgekehrt erst die Eigenvektoren x zu bestimmen und dann erst t, ist weniger glücklich, da er die Eigenvektoren aus einem linearen Gleichungssystem von $\frac{1}{2}(n^2-n)$ Unbekannten bekommt, welches aus $t=(Ax)/(Bx)$, $i=1,\cdots,n$ entsteht, wo A_i, B_i die i-te Zeile von A, B bezeichnet. Allein deren Lösung kostet schon etwa $n^6/24$ Multiplikationen. Am Schluss gibt Verf. einige Hinweise zur Konvergenz-Beschleunigung der gebräuchlichen Iterationsmethode. *E. Bodewig.*

Rubbert, F. K. Zur Praxis der numerischen Integration. Astr. Nachr. **277**, 161–166 (1949).

This is an expository paper giving the modern improvements on Adams' method of integrating differential equations numerically, and especially the convenient formulas introduced by Nyström, where a value is predicted by an open type quadrature formula (here expressed in differences) and corrected by the corresponding closed type formula. Likewise the method of Störmer for equations of the second order is presented, together with some new formulas which appear to be even more convenient than those of Störmer.

W. E. Milne (Corvallis, Ore.).

Huskey, Harry D. On the precision of a certain procedure of numerical integration. With an appendix by Douglas R. Hartree. J. Research Nat. Bur. Standards **42**, 57–62 (1949).

The cumulative effect of round-off errors in numerical solution of differential equations has been studied by Rademacher on the assumption that the round-off errors are randomly distributed. The paper by Huskey exhibits cases of actual numerical integration on the ENIAC which have very long runs of nonrandom round-off. The total effect of such runs can be quite significant. The appendix by Hartree investigates conditions under which such runs may be expected to occur. *W. E. Milne.*

Greenwood, Robert E. Numerical integration for linear sums of exponential functions. Ann. Math. Statistics **20**, 608–611 (1949).

A general method is presented for developing numerical integration formulas similar to the Newton-Cotes formula but giving exact answers for the integral of functions other than polynomials. This method is used to determine coefficients in two types of formulas where the integration of selected exponentials is exact. Estimates of error are given as well as seven examples integrated by several methods. It appears that the symmetrical exponential method compares favorably with the Newton-Cotes method for some functions. *S. Levy* (Washington, D. C.).

Schemmrich, Otto. Ein elektrisches Messverfahren zur Integral- und Mittelwertbestimmung. Arch. Elektrotechnik **34**, 415–423 (1940).

A conductance $G(t)$ is made proportional to the quantity of which the time integral is desired. The conductance $G(t)$ is shunted across a fixed condenser which is successively charged from a fixed source of potential E and then discharged through $G(t)$. The source E is turned on and off automatically by a glow lamp (neon bulb). The desired integral is then approximately given (error not studied) by a counter which records the total number of discharges. The basic idea is illustrated by a hydraulic analogy. Although physically $G(t)$ cannot drop below some constant G_0, it is shown that no limitation is involved if the number of impulses for G_0 is also observed. This makes it possible to integrate a function which becomes negative, a fact not explicitly mentioned by the author. The device is similar

in principle to the integrating exposure meter described by J. L. Michaelson [G. E. Review **42**, 92–93 (1939)] except that in the latter a photocell controls the current which charges the condenser and the resistance through which discharge takes place is constant. *R. Church.*

Beyerle, K. Der Schlupffehler des Reibradintegrators. Z. Angew. Math. Mech. **29**, 186 (1949).

The small friction load imposed by the counter-mechanism in a disc-and-wheel integrator produces an error by enabling the integrator to follow a line slightly inclined to a radius of the disc without registering the rotary component. In a test with a precision integrator measuring the area of a narrow right triangle lying near a radius, this error amounted to about 3 percent. *M. Goldberg* (Washington, D. C.).

Tomey, A. H. J. M. Une nouvelle méthode pour l'analyse des graphiques résultant de sinusoides amorties. Pont. Acad. Sci. Comment. **11**, 273–325 (1947).

En continuant la méthode de H. Labrouste et Y. Labrouste [voir Mém. Acad. Sci. Inst. France (2) **64**, no. 5 (1941); Analyse des Graphiques Résultant de la Superposition de Sinusoïdes, Presses Universitaires de France, Paris, 1943; ces Rev. **8**, 408] l'auteur indique une méthode pour déterminer exactement les composantes harmoniques ou anharmoniques d'une période fondamentale donnée de sorte que les variations continues de l'amplitude des composantes ressortent de l'analyse. Il donne un schéma pour trouver les 20 et un autre pour trouver les 40 premières sinusoides harmoniques d'un ordre donnée. Quelques figures terminent le mémoire. *E. Bodewig* (La Haye).

Pollak, L. W. Indirect autocorrelation method of searching for periodicities. Proc. Roy. Irish Acad. Sect. A. **52**, 143–161 (1949).

The author considers Fuhrich's method (that of repeated correlograms) for the detection of harmonic components in a series of values. The innovation is that the correlogram is calculated from the constants of a harmonic analysis. The advantages claimed are less labour, better checking, and the possibility of combining several available harmonic analyses of similar data. Numerical comparisons are given. *P. Whittle* (Uppsala).

Marsilii, Paolo. Un metodo per la risoluzione approssimata di problemi di calcolo operatoriale. Elettrotecnica **35**, 258–263 (1948).

Given the function $F(p)$ in $F(p)/p=\int_0^\infty e^{-pt}f(t)dt$, the author tries to obtain an approximate evaluation of $f(t)$. In $f(t)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}e^{pt}p^{-1}F(p)dp$ he replaces p by $1/a$, and writes $f(t)=-(2\pi i)^{-1}\int_C e^{t/a}F(1/a)a^{-1}da$, where the line of integration is a circle C through the origin, with its centre on the real axis. As all singularities of $F(p)$ are on the left of the straight line $(c-i\infty, c+i\infty)$ there are no singularities of $F(1/a)$ inside or on C. Now the function $F(1/a)$ is approximated by a polynomial $F^*(1/a)=\alpha+\beta a+\gamma a^2+\cdots+\nu a^n$. The coefficients $(\alpha, \cdots, \nu)$ are obtained by identification of $F(1/a)$ and $F^*(1/a)$ at $n+1$ arbitrarily chosen points a_k on or inside C. The approximate value of $f(t)$ is

$$f^*(t)=-(2\pi i)^{-1}\int_C e^{t/a}F^*(1/a)a^{-1}da.$$

Some examples are considered. By graphical integration an upper bound for the absolute value of the error is found. *S. C. van Veen* (Delft).

Schade, Th. Numerische Integration einer Differentialgleichung der Wellenmechanik. Z. Angew. Math. Mech. **29**, 210–224 (1949). (German. Russian summary)

The paper deals with the numerical solution of a special Schrödinger equation with independent variable r, where r is the distance from the origin in three-dimensional space. Two methods of solution are used, first a numerical integration over the range from $r=0$ to the last inflection point $r=r_w$, and second an asymptotic series to express the solution from $r=r_w$ to $r=\infty$. It is necessary to fit the two solutions together in the proper manner at $r=r_w$. A numerical example illustrates the method, and supports the author's contention that this method is shorter than those previously used. *W. E. Milne* (Corvallis, Ore.).

Solodovnikov, V. V. On the application of trapezoidal frequency characteristics to the analysis of the behavior of systems of automatic regulation. Avtomatika i Telemehanika **10**, 362–376 (1949). (Russian)

The author considers a differential equation of an automatic regulator and assumes that the solution of this equation by the Laplace transform method leads to the expression

$$(1)\qquad x(t)=2\pi^{-1}\int_0^\infty \omega^{-1}R(\omega)\sin t\omega\, d\omega,$$

where $x(t)$ is the unknown function and $R(\omega)$ (generalized real frequency characteristic) is a known function vanishing for sufficiently large values of ω. He proposes a method for a rapid determination of the properties of $x(t)$. This method consists of approximating $R(\omega)$ by a linear combination of "trapezoidal functions" $\tau(\omega)$ which are positive and constant for $0<\omega<\omega_1$, linear for $\omega_1<\omega<\omega_2$ and vanish for $\omega>\omega_2$. If $R(\omega)=\tau(\omega)$ the integral (1) can be expressed in closed form. For $\tau(0)=1$, $\omega_2=1$,

$$(2)\quad x(t)=2\pi^{-1}\{\mathrm{Si}\,(\kappa t)+(1-\kappa)^{-1}[\mathrm{Si}\,(t)-\mathrm{Si}\,(\kappa t)t^{-1}(\cos t-\cos \kappa t)]\},$$

where $\kappa=\omega_1/\omega_2$. Using this formula $x(t)$ may be found for any trapezoidal function $\tau(\omega)$. In particular, it is proposed to approximate $R(\omega)$ by a "typical function" which is a difference of two trapezoidal functions. The paper contains tables and graphs of the functions (2) and two examples. *L. Bers* (Princeton, N. J.).

Gehlshøj, Bent. Electromechanical and electroacoustical analogies and their use in computations and diagrams of oscillating systems. Ingeniørvidensk. Skr., Copenhagen **1947**, no. 1, 142 pp. (1947).

The two electrical analogues for mechanical systems are explained. For a number of cases, including filters and pick-ups of magnetic and crystal type, the construction of the analogous circuit is carried out. Vibrations of helical springs and the transverse and longitudinal vibrations of bars are discussed by finding their mechanical impedance; in the latter case the transmission line analogue is used. The basic circuit elements and laws for acoustical systems and their electrical analogues are explained. The acoustical impedance is determined for tubes and diaphragms. These results are used to discuss microphones and the telephone receiver. Several electromechanical transducers are studied in detail using the appropriate analogous electrical circuit. *R. Church* (Annapolis, Md.).

Gamburcev, G. A. On the existence of electromechanical analogies. Doklady Akad. Nauk SSSR (N.S.) **60**, 1335–1337 (1948). (Russian)

The author considers mechanical systems consisting of dashpots, springs and masses and electrical systems containing only lumped resistances, inductances and capacities. In a former paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **9** (1935 IV), 319–322] he explained a two-pole mass element and indicated briefly how, by regarding the plane of the drawing of the circuit as of infinite electrical or zero mechanical resistance, the analogue of either an electrical or a mechanical plane system (one capable of being drawn on a plane without crossings of branches) can be set up by placing each circuit element in a window in the plane, connecting the two poles of the element to opposite sides of the window, and then drawing suitable connecting lines for the electrical circuit and disconnecting lines perpendicular to them for the mechanical circuit. This argument serves to make it clear that every plane or space network has an analogue of the second kind (otherwise designated as an analogue on the mobility or electromagnetic basis) and every plane network also has an analogue of the first kind. It might have been expected that when a space system is drawn on a surface such that crossings are avoided (such as a torus) the method of the paper cited would establish an analogue of the first kind. Examples show this is not the case. In fact, the purpose of the present paper is to show that an analogue of the first kind does not exist for any electrical or mechanical space system. This is accomplished by showing first that a system does not have an analogue (first kind) if one of its subsystems does not (subsystems are obtained by setting the resistance of one or more elements equal to zero or infinity). Then the class of simplest space systems (having only plane subsystems) is examined and these systems are shown to have no analogue of the first kind.

Much of the argument must be supplied by the reader, including the exact definition of analogous systems. The result is not in conflict with methods for obtaining electrical analogues of the first kind for space systems described by A. Bloch [see the following review] since other kinds of circuit elements such as ideal transformers are not here considered. *R. Church* (Annapolis, Md.).

Bloch, A. On a method for the construction of networks dual to non-planar networks. Proc. Phys. Soc. **58**, 677–694 (1946).

Two networks are here said to be dual if there is a one-to-one correspondence between their elements and if the current through any element is proportional to the voltage across the corresponding element. It is well known that if the network is planar and contains only lumped resistances, inductances, capacities and sources of voltage or of current a dual is furnished by the topological dual of its graph. R. Julia [Bull. Soc. Française Électriciens **9**, 281–302 (1939)] has shown that if ideal transformers are permitted among the circuit elements, it is also possible to construct a dual for any nonplanar network. The author discusses Julia's method and gives modifications leading to four additional methods. Applications and practical considerations are indicated. *R. Church* (Annapolis, Md.).

McCann, G. D., Jr., and Wilts, C. H. Application of electric-analog computers to heat-transfer and fluid-flow problems. J. Appl. Mech. **16**, 247–258 (1949).

***Pantofíček, Jaroslav. Vyrovnávací Počet Statickou Methodou. [Adjustment of Observations by the Statical Method].** České Akademie Věd a Uměnf, Prague, 1949. 99 pp.

Already Gauss noticed the connection between the method of least squares and the equilibrium of mechanical systems. Suppose, for instance, that $x_1, \cdots, x_n$ are n direct measurements, with weights $p_1, \cdots, p_n$, respectively. If on a straight line we place concentrated masses, proportional to the p_i, at the points x_i, and if a variable point x is attracted by each of these masses, the potential energy being proportional to $(x-x_i)^2$, then the equilibrium position of x coincides with its value as given by the method of least squares. This connection has repeatedly been used, and in particular the graphical methods developed in the applications of statics to engineering problems can be exploited with great success in the practical solution of least-square problems.

In the first 5 chapters (38 pages) of this work, the author describes in general terms the application of statics, particularly of graphical statics, in this connection. The remainder of his work is devoted to a more detailed discussion of several specific problems, all arising in surveying. (VI) The four sides and two diagonals of a quadrilateral are measured (equal weights). (VII) The position of a point P (in the plane) is determined by measuring its distances from n given points, $P_1, \cdots, P_n$, and by measuring for each i the angle which PP_i makes with a fixed direction. (VIII) The position of P is determined by measuring its distances from n points (weights inversely proportional to distances). (IX) The position of P is determined by measuring angles. (X) Surveying of a polygon. In each case the author illustrates his methods by numerical examples.
A. Erdélyi (Pasadena, Calif.).

Ackerl, F. Der Vorwärtseinschnitt aus fehlerhaften Festpunkten. Österreich. Ing.-Arch. **3**, 235–241 (1949).

The author considers the following problem in surveying. An unknown point is determined by intersection. It is assumed that the coordinates of the two known points are subject to errors of measurement. The mean ellipse of error of the new point is determined. *E. Lukacs.*

Janko, Jaroslav. Difference equation of the policy reserve. Aktuárské Vědy **8**, no. 2, 68–75 (1949).

This is an expository article on a general form of life insurance with discontinuous payments. The author obtains the customary formulae for the premium reserve by considering the recurrence relation as a linear difference equation and determining the premium reserve as its solution.
E. Lukacs (Washington, D. C.).

MECHANICS

Dizioğlu, Bekir. Zur Kinematik getriebebeweglicher Kupplungen. Bull. Tech. Univ. Istanbul **1**, 11–28 (1943). (German. Turkish summary)

Grübler gave the following formulas for f (the number of degrees of freedom of a linkage mechanism) in terms of n (the number of links) and the f_s (the number of degrees of freedom of the separate joints): $f=3(n-1)-\sum(3-f_s)$ for plane linkages, $f=6(n-1)-\sum(6-f_s)$ for spatial linkages. This paper is an exposition of these formulas which are illustrated with various linkage mechanisms. Singular cases, which do not obey these formulas, are not considered.
M. Goldberg (Washington, D. C.).

MacColl, L. A. A theorem concerning the positions of equilibrium of a rigid body. Quart. Appl. Math. 7, 472–473 (1950).

Let B be a rigid body, of any form, which is free to rotate about a fixed point P. Let B be subjected to forces which are derived from a potential function V, which is an arbitrary single-valued function, of class C^2, of the coordinates of B. Then B has at least four distinct positions of equilibrium. The theorem is an immediate consequence of the following well-known theorems. (1) The configuration space of the rigid body is homeomorphic with real 3-dimensional projective space; (2) the number of distinct stationary points of a real single-valued function, of class C^2, defined over a closed manifold M is at least equal to the category of M; (3) the category of real n-dimensional projective space is $n+1$. *O. Bottema* (Delft).

Gran Olsson, R. Remarks on the motion of anchor chains. Quart. Appl. Math. 7, 463–470 (1950).

The problem of motion considered is explicitly solved in terms of a special case of the $\wp$- and zeta-functions of Weierstrass. For this case tables of the functions are available. *P. Franklin* (Cambridge, Mass.).

Hochrainer, A. Die elastische Aufstellung des starren Körpers. Österreich. Ing.-Arch. 3, 247–261 (1949).

This paper treats the essentially elementary problem of finding the vibration modes of small free and forced hormonic oscillations of a spring mounted rigid body. Tensor notation is employed. Two illustrative examples are given. The problem is a special case of one treated by Lagrange [Mécanique Analytique, part 2, section 5, 1st ed., 1788; part 2, section 6, 4th ed., 1888]. The reviewer is not convinced by the author's statement that by means of this procedure the modes of the system "can be calculated in a much clearer form than with methods used up to the present time." *E. Pinney* (Berkeley, Calif.).

Aminov, M. Š. On the stability of certain mechanical systems. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 643–646 (1948). (Russian)

The author considers a conservative mechanical system with n degrees of freedom subject to stationary holonomic constraints. The state of the system is determined by the coordinates $q^1, \cdots, q^n$, and the problem is treated as the motion along geodesics of the Riemannian manifold with metric $ds^2=g_{\alpha\beta}dq^\alpha dq^\beta$. Under special assumptions on the form of the $g_{\alpha\beta}$, and the corresponding $\vartheta_{\alpha\beta}$ for the perturbed system, certain sufficient conditions for stability are given. *E. A. Coddington* (Cambridge, Mass.).

Hydrodynamics, Aerodynamics, Acoustics

***Andjelitch, Tatomir. Les équations du mouvement d'un système non holonome dans un liquide incompressible.** Thesis, University of Belgrade, 1946. 51 pp. (Serbian. French summary)

Thomson's classic theorem [Lamb, Hydrodynamics, 6th ed., Cambridge University Press, 1932, chapter 6] on the motion of solids through a liquid is generalized for a nonholonomic system of solids. The Lagrange form of the Thomson equations is replaced by the Chaplygin-Woronetz form [Woronetz, Math. Ann. 70, 410–453 (1911)], everything else remaining the same. The proof follows the familiar formal pattern and starts with Woronetz' generalization of Hamilton's principle for nonholonomic constraints of the form $\dot{q}_{n+\nu}=\sum_1^n a_{\nu i}\dot{q}_i+a_\nu$. *A. W. Wundheiler.*

Magyar, F. Beitrag zur Feldtheorie der Flüssigkeitswirbel. Österreich. Ing.-Arch. 3, 241–246 (1949).

This paper contains one of the frequent rediscoveries and interpretations of Lamb's form of Bernoulli's theorem [Proc. London Math. Soc. (1) 9, 91–92 (1878); Hydrodynamics, 6th ed., Cambridge University Press, 1932, § 165]. *C. Truesdell* (Washington, D. C.).

Popov, S. G. On helical motions of an ideal fluid. Vestnik Moskov. Univ. 3, no. 8, 35–47 (1948). (Russian)

It is well known that the equations of helical motions (*) $\omega_x/u=\omega_y/v=\omega_z/w=\lambda(x, y, z)$ represent a special case of the Eulerian equations if the motion is steady, the external forces are conservative and the density of the fluid is a function of the pressure only. By considering helical motions in a cylindrical tube, on a sphere, in a ring-shaped tube and in a helical tube, the author gives examples of various types of helical vortices: straight vortex, vorticial sphere, vorticial ring and helical ring. As examples for applications of helical motions are mentioned the detonation spin in long tubes and the horseshoe vortices or more generally the free vortex lines in the case of variable circulation around an airwing of finite span. In both cases the vortex lines and the streamlines coincide, i.e., equations (*) are satisfied [Prandtl, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. 1918, 451–477]. Helical motions appear also in the problem of axisymmetrical flow and in the most general three-dimensional flow. *E. Leimanis.*

Lighthill, M. J. A note on cusped cavities. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2328 (9143), 3 pp. (1949).

An explicit construction of an incompressible plane potential flow past a symmetric profile, with the property that two free streamlines detach from the rear of the body to meet in a cusp. The author seems unaware of the work on such flows by Lavrentieff [Rec. Math. [Mat. Sbornik] N.S. 4(46), 391–458 (1938)] and Kolscher [Luftfahrtforschung 17, 154–160 (1940); these Rev. 4, 59]. *D. Gilbarg.*

Stipanić, Ernest. Sur une équation de l'hodographe de vitesse dans le plan et son application. Bull. Soc. Math. Phys. Serbie 1, 45–49 (1949). (Serbian. Russian and French summaries)

Sokolovskiĭ, V. V. On nonlinear filtration of ground water. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 525–536 (1949). (Russian)

The results of this paper were summarized earlier [Doklady Akad. Nauk SSSR (N.S.) 65, 617–620 (1949); these Rev. 10, 635]. Cf. also Christianovič [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 4, no. 1, 33–52 (1940)]. *E. Leimanis* (Vancouver, B. C.).

Polubarinova-Kočina, P. Ya. On transient filtration with a surface of separation. Doklady Akad. Nauk SSSR (N.S.) 66, 173–176 (1949). (Russian)

Polubarinova-Kočina, P. Ya. On the transient motion of ground water in filtration from reservoirs. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 187–206 (1949). (Russian)

Vedernikov, V. V. Filtration in the presence of a draining or a water-carrying layer. Doklady Akad. Nauk SSSR (N.S.) **69**, 619–622 (1949). (Russian)

van Deemter, J. J. Results of mathematical approach to some flow problems connected with drainage and irrigation. Appl. Sci. Research A. **2**, 33–53 (1949).

Casal, Pierre. Mouvement permanent d'un fluide visqueux entre deux disques en rotation. C. R. Acad. Sci. Paris **230**, 178–179 (1950).

Ray, M. Development of liquid motion due to an impulse. Bull. Calcutta Math. Soc. **41**, 179–182 (1949).

The particles in the plane $y=0$ of a viscous fluid of infinite extent are impulsively given a velocity $u(x)$. The resulting motion throughout the fluid is then computed by a successive approximation scheme based on the boundary layer equations, with nonlinear terms omitted in the initial approximation. The first two terms of the result are quoted explicitly for $u(x)=e^{-x}$. *G. F. Carrier* (Providence, R. I.).

Viguier, G. Les forces tangentielles de viscosité avec gradients de vitesse élevés. Experientia **5**, 397–398 (1949).

The author considers the dependence of the viscosity coefficient on the shear when it is large in order to explain the relation of skin friction to Reynolds number for flow over a flat plate. *C. C. Lin* (Cambridge, Mass.).

Howarth, L. Rayleigh's problem for a semi-infinite plate. Proc. Cambridge Philos. Soc. **46**, 127–140 (1950).

This paper is concerned with the unsteady flow of an incompressible viscous fluid, set up by a semi-infinite plate coincident with the half plane $x\geqq 0$, $y=0$; initially, the plate is at rest and is made suddenly to move with a constant velocity W parallel to the z-axis. The problem is solved by assuming the flow to be one-dimensional. With this simple assumption, a solution for the exact equation is developed in two different forms: series form in polar coordinates and integral form in parabolic coordinates. These two forms of solution give an adequate means for evaluating the velocity-distribution and the skin friction. The numerical result shows that the edge effect on the skin friction is confined to a region of the order of a boundary layer thickness. Following Rayleigh's hypothesis, the skin friction coefficient for a semi-infinite plate of large finite breadth, in a steady stream normal to its leading edge, is deduced. *Y. H. Kuo* (Ithaca, N. Y.).

***Purday, H. F. P. An Introduction to the Mechanics of Viscous Flow. Film Lubrication, the Flow of Heat by Conduction, and Heat Transfer by Convection. ("Streamline Flow").** Dover Publications, Inc., New York, N. Y., 1949. vi+185 pp. $2.75.

This book appears to be of primary interest to workers in problems of lubrication and heat transfer. Turbulent flow is omitted, as implied in the title of the book. In the presentation of the material the author intends to show "general principles at work in their simplest forms leading quickly to concrete useful results." Accordingly, nearly the whole book is devoted to discussion of specific problems of practical interest. Only in the last chapter are the fundamental equations of fluid flow derived. The mathematical technique used is at a level usually included in a course of advanced calculus for engineers, but such knowledge is not assumed in advance. The titles of various chapters will serve to explain the above points more fully: (I) Introduction, (II) Viscous flow in pipes and channels, (III) Viscous flow between fixed parallel planes, (IV) Films of varying thickness, (V) Sliding pads, (VI) Journal bearings, (VII) Viscous resistance of bodies, (VIII) The flow of heat, (IX) Boundary convection, (X) Convection in depth, (XI) Orthogonal functions, (XII) Equations of flow.

The reviewer cannot help regretting that basic considerations and systematic investigations have not been sufficiently emphasized in this book. For example, in chapter I, it is stated that the stress in a viscous fluid is proportional to the velocity gradient, without a full discussion of shear strain (not even in the last chapter). This could be misleading (e.g., in the case of flow between rotating cylinders). Again, in the discussion of the boundary layer theory, the von Kármán-Pohlhausen approximation (which may be regarded as the beginning step of a systematic treatment) could have been included instead of the even cruder approximation used in the text. In general, the treatment in the book is sound, and it ought to have its place among books on the mechanics of viscous flow. *C. C. Lin.*

Lin, C. C. Note on the characteristics in unsteady one-dimensional flows with heat addition. Quart. Appl. Math. **7**, 443–445 (1950).

The theory of characteristics described in the title is important in the theory of jet propulsion, as the author notes. He obtains a practical method of calculation by eliminating density from the equations of flow, getting a set of equations having derivatives of only two variables, the pressure and the velocity. These readily lend themselves to a standard method for getting the equations of the characteristic curves and the compatibility equations. The author shows how the latter may be integrated to find the solution throughout a region of the space-time plane from a knowledge of the solution along a curve in that plane. *E. Pinney* (Berkeley, Calif.).

Carrier, G. F., and Lewis, J. A. On heat transfer problems in viscous flow. Quart. Appl. Math. **7**, 450–457 (1950).

The temperature distribution in a two-dimensional flow field of incompressible viscous fluid through channels is considered from a rather general point of view. By defining a characteristic parameter ϵ which depends on the fluid properties, the velocity of the flow and the geometry of the channel, the problems are classified into three distinct categories, according to $\epsilon>1$, $\epsilon\approx 1$ and $\epsilon<1$. In the case of $\epsilon>1$, namely, large conductivity or small channel width, the temperature distribution is uniform across the channel. When ϵ is of order unity, the mathematical problem is difficult and one has to rely on numerical methods. In the case of $\epsilon<1$, the boundary conditions require a boundary-layer type of solution. This case is discussed in some detail and a specific example is given. *Y. H. Kuo.*

Fainzil'ber, A. M. Thermal loss of energy in flows of a viscous fluid at large Reynolds numbers. Doklady Akad. Nauk SSSR (N.S.) **69**, 503–506 (1949). (Russian)

The author starts with the differential equations of a viscous fluid for large Reynolds numbers. Into these he introduces the dimensionless quantities

$$y_1=[\bar{u}(x)-u(x,y)]/\bar{u}(x), \quad x_1=x/x_m$$

as independent variables and the relative loss of energy $E_1=2\mu(\partial u/\partial y)^2/(\rho\bar{u}_k^2)$ as the dependent dimensionless variable. The equation for E_1 is then solved in the series form $E_1=\sum_{i=2}^{n}B_i(x_1)y_1^{(i)}$ (m and k are some characteristic points;

This paper contains a part of the author's doctoral dissertation dated Jan., 1948. His treatment of simple waves altogether avoids use of the theory of characteristics.

for example, m may be taken as the point of minimum pressure; both points may coincide). The new variables are found to be suitable for both finite and asymptotic layers. The distribution of velocities and the distribution of dimensionless temperature are described using the known E_1. Finally the author considers the special case when the free stream velocity distribution is prescribed to be $\bar{u}(x_1)=(\alpha x_1+\beta)^r$ (power distribution of velocities).

E. Leimanis (Vancouver, B. C.).

Gutman, L. N. On laminar thermal convection above a stationary heat source. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 435–444 (1949). (Russian)

On the basis of the equations of free convective laminar flow, the flow pattern established by a stationary heat source is considered. It is shown that the radial and vertical velocity components and the temperature are given respectively by:

$$u=2\mu r^{-1}(\eta F'-F),\quad w=(\lambda Q/\mu)^{\frac{1}{2}}F',\quad \theta=Q(\mu z)^{-1}F'',$$

where μ is the viscosity, λ is the coefficient of thermal expansion of air, Q is the strength of the heat source, and F is the solution of $\eta F'''+FF''=0$ with the boundary conditions $F(0)=0$, $F'(0)=1$, $F'(\infty)=0$ and $\eta=r^2(4z)^{-1}(\lambda Q/\mu^2)^{\frac{1}{4}}$. Tables are given for F and its derivatives for

$$\eta=0(.1)3(.5)10(1)20(10)100.$$

The analysis is extended to a heat source in a vertical moving column of air with particular attention to the two limiting cases of large and small vertical velocities.

N. A. Hall (Minneapolis, Minn.).

Kalikhman, L. E. Gasdynamic theory of heat. Tech. Rep. no. **F-TS-1211-IA (GDAM A9-T-30)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+51 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 449–474 (1946); these Rev. **8**, 418; for another translation see Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1229** (1949); these Rev. **10**, 645.

Cap, F. Über eine Kopplung von Strömungs- und Verbrennungsvorgängen. Österreich. Ing.-Arch. **3**, 97–106 (1949).

The author has made an attempt to modify the principal equations of aerodynamics in such a way as to account for the accompanying combustion. By using the Charbonnier-Schmitz-Vieille law of combustion the equation for the characteristics of the potential equation is obtained in the form

$$\frac{du}{dt}\pm\frac{2}{u-1}\frac{da}{dt}=\rho^*\frac{du}{dt}-q(u-u^*)\pm\tilde{q}a.$$

Under certain assumptions a graphical procedure based on this equation leads to the solution of nonsteady one-dimensional aerodynamical problems of flow accompanied by combustion (rockets,, cannons, ram-jets) [cf. Cap, Acta Physica Austriaca **1**, 89–97 (1947); these Rev. **9**, 216].

E. Leimanis (Vancouver, B. C.).

***Ehlers, F. E. Methods of linearization in compressible flow. III. Prandtl-Glauert method.** Prepared under the supervision of G. F. Carrier. Tech. Rep. no. **F-TR-1180C-ND (A-9-M-IV/3)**. Headquarters Air Materiel Command, Wright Field, Dayton, Ohio. v+141 pp. (13 plates) (1948).

[For parts I and II see the same Rep. nos. **F-TR-1180A-ND (GDAM A-9-M IV/I), F-TR-1180 B-ND (GDAM A-9-M IV/II)** (1948); these Rev. **10**, 161, 753.] Except for the theory of lifting surfaces and wing-body interference in supersonic flow, this monograph contains an account of the whole literature, up to 1949, on small disturbances created by slender bodies in a uniform compressible free stream, the so-called Prandtl-Glauert method. *H. S. Tsien.*

Cabannes, Henri. Écoulement potentiel discontinu d'un fluide parfait compressible. C. R. Acad. Sci. Paris **229**, 510–511 (1949).

Pairs of nonuniform plane steady potential air flows, separated from each other by a shock, are considered in general; the shock needs to be of uniform strength. The discussion is all too brief, but includes mention, as an example, of a case where both flows are of Tollmien's spiral form. *M. J. Lighthill* (Manchester).

Ghaffari, A. A simplified theory of "simple waves." Aeronaut. Quart. **1**, 187–194 (1949).

This paper integrates the equations of plane irrotational flow of a perfect gas in the case when the hodograph is degenerate. The results are well known, at least in the United States, and are discussed from a more general point of view by Courant and Friedrichs [Supersonic Flow and Shock Waves, Interscience, New York, 1948, §§ 109–112; these Rev. **10**, 637] although some of the author's explicit formulae are not written down there. *C. Truesdell.*

Coburn, N. Degenerate two-dimensional non-steady irrotational flows of a compressible gas. Quart. Appl. Math. **7**, 439–443 (1950).

Non-steady, two-dimensional, irrotational, compressible flow is degenerate if it contains two distinct families of real cylindrical characteristic surfaces with generators parallel to the t-axis. A necessary and sufficient condition for degeneracy is that the flow be supersonic, the Cartesian velocity components $v_\lambda=f(t)\bar{v}_\lambda(x^1,x^2)$ $(\lambda=1,2)$, and the velocity of sound $c=f(t)\bar{c}(x^1,x^2)$, where $-1/f=kt+b$ for some constants b,k. None of these degenerate flows is a simple wave, i.e., a flow in which all curves $\bar{v}_1=$constant are both curves $\bar{v}_2=$constant and straight characteristics in the (x^1,x^2)-plane. There exist degenerate flows in which all streamlines are logarithmic spirals. *J. H. Giese.*

Ovsyannikov, L. V. On a gas flow with a straight transition line. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 537–542 (1949). (Russian)

The author claims to have obtained a solution of the differential equation of a two-dimensional gas flow which describes a subsonic flow in a symmetrical jet which becomes sonic along a straight line L perpendicular to the axis of the jet. The streamlines intersect L at right angles. The solution is obtained as a limiting case of Chaplygin's well-known solution describing a purely subsonic jet [S. Chaplygin, Učenye Zapiski Imp. Moskov. Univ., Otd. Fiz. Mat. **21**, 1–121 (1904); translated as Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1063** (1944); these Rev. **7**, 495].

L. Bers (Princeton, N. J.).

Prim, R. C. A new superposition principle for steady gas flows. Quart. Appl. Math. **7**, 445–450 (1950).

Defining the reduced velocity **W** as $\mathbf{V}/a$, where a is the ultimate velocity magnitude attainable on a given streamline by expansion to zero pressure [M. Munk and R. Prim, Proc. Nat. Acad. Sci. U. S. A. **33**, 137–141 (1947); R. Prim, J. Appl. Phys. **20**, 448–450 (1949); these Rev. **10**, 752],

the author asks, given a plane flow field with reduced velocity $\mathbf{W}_p$, what spatial reduced velocity field $\mathbf{W}$ can be obtained by superimposing a uniform velocity field normal to the given plane flow field. If $\mathbf{k}$ is the unit vector normal to the plane flow, the answer is $\mathbf{W}=(1-W_n^2)^{\frac{1}{2}}\mathbf{W}_p+\mathbf{k}W_n$, where W_n is an arbitrary function, constant on any streamline of the plane flow, of one sign and $|W_n|\leqq 1$. The author then shows the "generalized Beltrami flows" [P. Neményi and R. Prim, Proc. Nat. Acad. Sci. U. S. A. **34**, 119–124 (1948); these Rev. **9**, 476] can be generated by this method of superposition. *H. S. Tsien* (Pasadena, Calif.).

Miles, John W. A note on supersonic flow in the Trefftz plane. Quart. Appl. Math. **7**, 470–472 (1950).

Beginning with a Fourier-integral expression for the potential $\phi(x, y, z)$ due to a supersonic doublet distribution $\gamma(x, y)$ in the plane $z=0+$, the author obtains

$$\phi(\infty, y, z)=(z/\pi)\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}d\eta\frac{\gamma(\xi,\eta)}{z^2+(y-\eta)^2}.$$

In the plane $z=0+$ this reduces to $\int_{-\infty}^{\infty}\gamma(\xi, y)d\xi$.

W. R. Sears (Ithaca, N. Y.).

Kaplan, Carl. On similarity rules for transonic flows. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **894**, 5 pp. (1948).

Formerly issued as Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1527** (1948); these Rev. **9**, 477.

Kudriashov, L. K. Plane parallel gas flow past an ellipse. Tech. Rep. no. **F-TS-1208-IA (GDAM A9-T-25)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+20 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 119–128 (1947); these Rev. **10**, 162.

Kiselev, B. M. Calculation of one-dimensional gas flows. Tech. Rep. no. **F-TS-1209-IA (GDAM A9-T-27)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+31 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 177–192 (1947); these Rev. **10**, 74.

Frankl, F. I. Asymptotic resolution of Chaplygin's functions. Tech. Rep. no. **F-TS-1212-IA (GDAM A9-T-31)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+7 pp. (1949).

Translated from Doklady Akad. Nauk SSSR **58**, 757–760 (1947); these Rev. **10**, 215.

Frankl, F. I. The flow of a supersonic jet from a vessel with plane walls. Tech. Rep. no. **F-TS-1213-IA (GDAM A9-T-32)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+7 pp. (1949).

Translated from Doklady Akad. Nauk SSSR **58**, 381–384 (1947); these Rev. **10**, 161.

Eterman, I. I. Pressure distribution on bodies of revolution in high subsonic gas flows. Tech. Rep. no. **F-TS-1222-IA (GDAM A9-T-41)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+17 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 363–370 (1947); these Rev. **9**, 543.

Falkovich, S. V. A class of de Laval nozzles. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1236**, 15 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 223–230 (1947); these Rev. **9**, 390.

Falkovich, S. V. Plane motion of a gas at hypersonic velocity. Tech. Rep. no. **F-TS-1221-IA (GDAM A9-T-40)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+11 pp. (1949).

Falkovich, S. V. Two-dimensional motion of a gas at large supersonic velocities. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1239**, 10 pp. (1949).

These are two independent translations from Akad. Nauk SSSR. Prikl. Mat. Meh. **11**, 459–464 (1947); these Rev. **9**, 476.

Falkovich, S. V. The lift force of a wing of finite span in supersonic flow. Tech. Rep. no. **F-TS-1205-IA (GDAM A9-T-23)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+12 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 171–176 (1947); these Rev. **9**, 543. Previously issued as Brown Univ. Grad. Div. Appl. Math. Translation **A9-T-23**.

Christianovich, S. A., and Falkovich, S. V. Three papers on supersonic flow. Amer. Math. Soc. Translation no. **10**, 35 pp. (1950).

Translated from Christianovich, Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 215–222 (1947); these Rev. **9**, 390; Falkovich, the two papers quoted above.

Falkovich, S. V. On the theory of a wing of finite aspect ratio in supersonic flow. Tech. Rep. no. **F-TS-1215-IA (GDAM A9-T-34)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+9 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 391–394 (1947); these Rev. **9**, 477.

Haskind, M. D., and Falkovich, S. V. An oscillating wing of finite aspect ratio in supersonic flow. Tech. Rep. no. **F-TS-1219-IA (GDAM A9-T-38)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+13 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 371–376 (1947); these Rev. **9**, 114.

Gurevich, M. I. Lift force of an arrow-shaped wing. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1245**, 14 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 513–520 (1946); these Rev. **8**, 415.

Gurevich, M. I. Remarks on the triangular wing in supersonic flow. Tech. Rep. no. **F-TS-1214-IA (GDAM A9-T-33)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+10 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 297–300 (1947); these Rev. **9**, 115.

Gurevich, M. I. On the problem of a thin triangular wing moving with supersonic velocity. Tech. Rep. no. **F-TS-1216-IA (GDAM A9-T-35)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton Ohio. ii+5 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 395–396 (1947); these Rev. **9**, 114.

Galin, L. A. A wing of rectangular plan form in supersonic flow. Tech. Rep. no. **F-TS-1217-IA (GDAM A9-T-36)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+20 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 465–474 (1947); these Rev. **9**, 254.

Galin, L. A. Remarks on the wing of finite aspect ratio in supersonic flow. Tech. Rep. no. **F-TS-1218-IA (GDAM A9-T-37)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+9 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 383–386 (1947); these Rev. **9**, 254.

Goldstein, Arthur W., and Jerison, Meyer. Isolated and cascade airfoils with prescribed velocity distribution. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **869**, 15 pp. (1947).

An exact solution of the problem of designing an airfoil with a prescribed velocity distribution on the suction surface in a given uniform flow of an incompressible perfect fluid is obtained by replacing the boundary of the airfoil by vortices. The procedure of finding the shape begins by choosing an approximate shape and distributing the vortices on it. The vortex strength per arc length at any point is equal to the magnitude of the velocity at that point, which is assumed specified. The stream function of the flow induced by the distribution of vortices and the uniform stream is then computed at points on the boundary of the assumed shape. If this stream function is constant the assumed shape is correct. Variations of the stream function are a measure of the deviation of the assumed shape from the correct one. The variations are used to distort the original shape into a new shape whose stream function is more nearly constant, the process being repeated until the variations become negligible. In the process of shape adjustment the velocity is altered on the pressure surface. The method is applicable to both isolated airfoils and to airfoils in cascade. The conformal transformation of the designed airfoil into a circle can then be obtained and the velocity distribution at any angle of attack computed. Numerical illustrations of the method are given for the airfoil in cascade, the computations being executed graphically.

R. M. Morris (Cardiff).

Thiruvenkatachar, V. R. Compressible shear flow. I. Thin airfoil in steady supersonic flow. Proc. Nat. Inst. Sci. India **15**, 219–231 (1949).

The problem is that of steady flow over a thin airfoil near the X-axis when the undisturbed velocity is in the X-direction and is a linear function of y, the shear flow. The velocity of sound in the undisturbed flow is taken to be a constant. Elementary solutions of the linearized equations of motion near the X-axis are first obtained when the Mach number near the X-axis is greater than unity. These solutions are then superimposed to obtain the solution for the airfoil. From these, the lift and drag of an airfoil are calculated in terms of the airfoil profile. In the opinion of the reviewer, the author's choice of elementary solutions for the airfoil problem is arbitrary as he has not considered the behavior of the solution far away from the airfoil. Until this is checked, the results cannot be accepted.

H. S. Tsien (Pasadena, Calif.).

Thiruvenkatachar, V. R. Compressible shear flow. II. Nonsteady motion of thin airfoil in supersonic stream. Proc. Nat. Inst. Sci. India **15**, 233–240 (1949).

The author gives an integral formula for the linearized velocity potential of two-dimensional unsteady shear flow over a thin airfoil by a technique developed by Taussky [Courant Anniversary Volume, Interscience Publishers, New York, 1948, pp. 421–435; these Rev. **9**, 392] for the case of uniform undisturbed flow, i.e., by using the elementary solution. However, as in the author's previous paper [preceding review], the result is doubtful since he has not considered the conditions at infinity. *H. S. Tsien.*

Miles, John W. A formulation of the aeroelastic problem for a swept wing. J. Aeronaut. Sci. **16**, 477–490 (1949).

Miles, John W. On nonsteady motion of delta wings. J. Aeronaut. Sci. **16**, 568–569 (1949).

The velocity potential $\phi(x, y, 0+)$ for an oscillating supersonic delta wing is written down according to Garrick and Rubinow [Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1383** (1947); these Rev. **9**, 216]. It is shown that certain spanwise weighted integrals of ϕ can be reduced to integrals arising in the two-dimensional problem. These are, in particular, the weighted integrals for lift and rolling moment on a spanwise strip. They are brought into the form of integrals involving the Green's function of a plane flow. A similar device can be used when the motion is not a steady oscillation but a motion beginning at an instant $t=0$; in this case the Green's function is replaced by its Bromwich integral. *W. R. Sears* (Ithaca, N. Y.).

Garrick, I. E., and Rubinow, S. I. Theoretical study of air forces on an oscillating or steady thin wing in a supersonic main stream. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **872**, 14 pp. (1947).

Formerly issued as Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1383** (1947); these Rev. **9**, 216.

Reissner, Eric. On the theory of oscillating airfoils of finite span in subsonic compressible flow. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1953**, 37 pp. (1949).

The oscillation of a wing of finite span in subsonic compressible flow is treated by lifting surface theory. The boundary value problem is formulated in terms of the velocity potential ϕ, and its solution reduced to that of an integral equation of great complexity, involving four integrals, of which two are double and contain derivatives of the unknown functions (of which one is of two variables and the other of one). For large aspect ratio the integral equation is approximated by one simpler in that the double integrals are replaced by single integrals. The author does not claim that the work is of use for calculating how an oscillating wing of given shape will behave. He lays stress rather on the fact that his final equation separates the effects expected of a two-dimensional aerofoil and of a lifting line. *M. J. Lighthill* (Manchester).

Reissner, Eric. Boundary value problems in aerodynamics of lifting surfaces in non-uniform motion. Bull. Amer. Math. Soc. **55**, 825–850 (1949).

This lecture is an account of some of (with references to all of) the mathematical methods which have been employed to study wing oscillation by lifting surface theory, including the two-dimensional case, and with Mach number either negligibly small, subsonic, or supersonic.

M. J. Lighthill (Manchester).

Rott, N. Transient phenomena at sonic speed. J. Aeronaut. Sci. **16**, 439–440 (1949).

Using his results [same J. **16**, 380–381 (1949); these Rev. **10**, 754] for the oscillating airfoil at a Mach number M of unity, the author calculates the corresponding case of entry into a sharp gust. The results are found to be the same as are obtained by letting $M \to 1$ in the analogous supersonic results [Miles, same J. **15**, 592–598 (1948); these Rev. **10**, 411]. *W. R. Sears* (Ithaca, N. Y.).

Hunter-Tod, J. H. The aerodynamic derivatives with respect to rate of yaw for a delta wing with small dihedral at supersonic speeds. Coll. Aeronaut. Cranfield. Rep. no. **28**, 16 pp. (4 plates) (1949).

Jaeckel, K. Vereinfachte Herleitung der Weissingerschen Zirkulationsgleichung für den schiebenden Tragflügel. Veröffentlichungen Math. Inst. Tech. Hochschule Braunschweig **1947**, no. 8, i+5 pp. (1947).

Reconsidering the problem of a yawed lifting line, this author notes that the downwash induced just above the line can be written as the sum of five terms, of which one vanishes and only one becomes infinite (logarithmically) at the line. He proposes to calculate this singular term approximately by distributing the corresponding increment of circulation over the chord in a certain fashion and using plane-flow formulas. This results, finally, in an integral equation for the total circulation distribution that is the same as Weissinger's [Math. Nachr. **2**, 45–106 (1949); these Rev. **11**, 64; also Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1120** (1947); these Rev. **8**, 542] although the derivation is different. [For another point of view, see Reissner, Proc. Nat. Acad. Sci. U. S. A. **35**, 208–215 (1949); these Rev. **10**, 753.] *W. R. Sears* (Ithaca, N. Y.).

Legras, Jean. Efforts aérodynamiques sur un profil en mouvement oscillant amorti. C. R. Acad. Sci. Paris **229**, 699–700 (1949).

Unsteady motion of a thin airfoil of infinite span in an incompressible perfect fluid is considered. The potential and vortex-sheet strength, as well as the total circulation $\Gamma(t)$ can be calculated in two parts, of which the first neglects the effects of the vortex wake while the second represents these effects. The relationship between $\Gamma(t)$ and the wake distribution then suffices to give an integral equation for $\Gamma(t)$. In the case of steady oscillations this is solvable in Bessel functions. [Reviewer's note: a similar method was used by von Kármán and Sears, J. Aeronaut. Sci. **5**, 379–390 (1938).] *W. R. Sears* (Ithaca, N. Y.).

Jones, Robert T. Subsonic flow over thin oblique airfoils at zero lift. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **902**, 7 pp. (1948).

Formerly issued as Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1340** (1947); these Rev. **8**, 542.

Cohen, Doris. Theoretical loading at supersonic speeds of flat swept-back wings with interacting trailing and leading edges. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1991**, 55 pp. (1949).

Roper, Gwendolen M. The yawed delta wing at incidence at supersonic speeds. Quart. J. Mech. Appl. Math. **2**, 354–373 (1949).

In an earlier paper [same Quart. **1**, 327–343 (1948); these Rev. **10**, 338] the author showed that the velocity components u, v, w of the linearized theory satisfy the equations

$$P_{xx}+P_{yy}=0, \quad P_{xx}-P_{yy}=0$$

inside and outside the nose Mach cone, respectively, where $\theta=\tan^{-1}(z/y)$, $\operatorname{sech}\sigma=\beta(y^2+z^2)^{\frac{1}{2}}/x=\sec\chi$. Here x, y, z are rectangular Cartesian coordinates, x measured in the direction of the stream, whose Mach number is M, and $\beta^2=M^2-1$. For a yawed delta wing whose leading edges are both inside the cone, a transformation is used that maps the region concerned into a rectangle. Results obtained by Stewart [Quart. Appl. Math. **4**, 246–254 (1946); these Rev. **8**, 109] for the unyawed wing can then be used. For the case when both leading edges are outside the cone, the solution for the inside region is determined in the plane of $\zeta=-\cos(\theta+i\sigma)$. The exterior solutions are known. For the case when one edge lies inside and one outside, a somewhat different transformation is used and the interior region is mapped in a semi-infinite rectangle. In all three cases the lift is calculated. The results agree with the appropriate results for unyawed wings. Numerical results are given for a certain isosceles delta wing. *W. R. Sears* (Ithaca, N. Y.).

Riegels, F. Das Umströmungsproblem bei inkompressiblen Potentialströmungen. II. Ing.-Arch. **17**, 94–106 (1949).

[For part I cf. Ing.-Arch. **16**, 373–376 (1948); these Rev. **10**, 490.] It is proposed to treat the flow about an arbitrary profile by distributing singularities along an axis lying within the profile. This technique is commonly used in the so-called thin airfoil theory. The present paper represents an attempt to extend it to thicker shapes. The formulas are derived by a kind of successive-approximation procedure and their accuracy is not clear. Force and moment are finally calculated, a procedure for numerical computation is set up, and tables of coefficients are given. A number of examples are treated, including ellipse, circular arc, and symmetrical Joukowski airfoil. In some of these cases the results agree in detail with the exact results obtainable by conformal mapping. In others the results resemble thin-airfoil results, having, for example, singularities at the leading edge. *W. R. Sears* (Ithaca, N. Y.).

Heaslet, Max. A., and Lomax, Harvard. The use of source-sink and doublet distributions extended to the solution of boundary-value problems in supersonic flow. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **900**, 15 pp. (1948).

Formerly issued as Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1515** (1948); these Rev. **9**, 312.

Heaslet, Max. A., Lomax, Harvard, and Jones, Arthur L. Volterra's solution of the wave equation as applied to three-dimensional supersonic airfoil problems. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **889**, 22 pp. (1947).

Formerly issued as Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1412** (1947); these Rev. **9**, 114.

Smith, C. B. A solution for the lift and drag of airfoils with air inlets and suction slots. J. Aeronaut. Sci. **16**, 581–589 (1949).

The additional lift and drag on aerofoils due to suction through slots is calculated (on potential theory) in terms of the position of these slots on a circle onto which the aerofoil is conformally mapped. The answers are graphically displayed for several special cases. *M. J. Lighthill.*

Perl, W., and Moses, H. E. Velocity distributions on two-dimensional wing-duct inlets by conformal mapping. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **893**, 13 pp. (1948).

Mirels, Harold, and Haefeli, Rudolph C. The calculation of supersonic downwash using line vortex theory. J. Aeronaut. Sci. **17**, 13–21 (1950).

Tomotika, S., and Tamada, K. Studies on two-dimensional transonic flows of compressible fluid. I. Quart. Appl. Math. **7**, 381–397 (1950).

By introducing a new variable $w=\int_1^q \rho q^{-1}dq$, q and ρ being respectively the normalized speed and density of the gas, the fundamental differential equation for w in the (φ, ψ)-plane can be approximated by $\partial^2 w/\partial^2\psi=\partial^2[\frac{1}{2}(\gamma+1)w^2]/\partial^2\varphi$ in the transonic region, γ being the ratio of the specific heats. A group of exact solutions have been found and the nature of the flows associated with each of them is mentioned. The most important result achieved in this connection is the one which embodies both Taylor's and Meyer's types of flows between two symmetrical curved walls in one solution depending on a parameter. By varying the value of this parameter, a family of Taylor's type and an isolated flow of Meyer's type are obtained. It is shown for the first time that Meyer's type of solution cannot be approached in a continuous manner from Taylor's type.

Y. H. Kuo (Ithaca, N. Y.).

Schaefer, M. Formation of envelopes of curved Mach waves in flow along a convex wall. Tech. Rep. **F-TS-1203-IA (GDAM A9-T-13)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. iii+44 pp. (1949).

Schaefer, M. The appearance of a compression shock in the neighborhood of a convex wall surface which is free from singularities. Tech. Rep. **F-TS-1206-IA (GDAM A9-T-14)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+22 pp. (1949).

[Translated from Technische Hochschule Dresden. Peenemunde Archiv **44/12a, 44/12b** (1944).] The problem investigated is whether an envelope of characteristics (limit line) can occur near a convex wall which has two singularities. The author's method is to assume the existence of the envelope, or at least a small segment of it, in the fluid, and to determine by the theory of characteristics that a convex regular streamline may exist near it. Both families of characteristics are involved in this calculation. The wall is found to have rather gentle curvature, diminishing gradually downstream. The flow pattern is continued upstream to the sonic line but not beyond, so the question of imbedding this supersonic region in a subsonic flow over a body is not solved. The first report gives the complete mathematical analysis, while the second is a summary report, not complete in mathematical detail, but giving more information about the wall shape, the continued flow, etc. The translation from the German is not quite free of ambiguities. Incidentally, four important diagrams, which were missing from the second of the original reports, are missing from the translation as well. *W. R. Sears.*

***Laporte, O., Bartels, R. C. F., and O'Rourke, R. C. An Investigation of the Exact Solutions of the Linearized Equations for the Flow Past Conical Bodies. Part III. Supersonic Flow Past an Elliptic Cone at an Angle of Attack.** Engineering Research Institute, University of Michigan, CM 575. Ann Arbor, Mich., 1949. iv+105 pp.

The steady supersonic airflow past an elliptic cone is treated by using the linearised equation of homentropic flow. Thus the work will be a good approximation when all the generators of the cone make an angle with the stream direction which is small compared with the Mach angle. The boundary conditions are satisfied exactly. This leads to a rather complicated solution in terms of elliptic functions, instead of the simple solution [Ward, Quart. J. Mech. Appl. Math. **2**, 75–97 (1949); these Rev. **10**, 644] which is available if the same approximation is made to the boundary conditions as is made in deriving the differential equation. The solutions are displayed in numerous tables and graphs.

M. J. Lighthill (Manchester).

Berman, Ya. R. The separation of streamlines from a circular cylinder in a bounded stream. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 543–546 (1949). (Russian)

The author extends the method used by Brodetsky [Proc. Roy. Soc. London. Ser. A. **102**, 542–553 (1923)] for approximating the discontinuous flow about a circular cylinder in an infinite fluid to the case where the cylinder is situated midway between two parallel walls. A graph is given for the drag coefficient and the angle of separation as functions of the ratio of cylinder diameter to canal width. When the formulas are interpreted in terms of cavitation number they seem to show good agreement with experiment.

J. V. Wehausen (Falls Church, Va.).

Astaf'ev, V. M. The differential equations of gas turbines with an infinite number of blades and their integrals. Doklady Akad. Nauk SSSR (N.S.) **68**, 449–452 (1949). (Russian)

Curvilinear coordinates, designed for the description of flows in turbines with infinitely many blades (vanes) of zero thickness, are introduced in three space, by means of any three real-valued functions r, φ, and s, having suitable properties. The level surfaces of r (the stream surfaces of a flow) are surfaces of revolution about the x-axis; the level surfaces of φ (the stationary position of the blades of the turbine, or the successive positions of the blades if they are rotating about the x-axis) are obtainable from each other by (rigid) rotation about the x-axis; and finally the level surfaces of s (the cross sections of the turbine) are determined from any given surface S of revolution about the x-axis (distinct from all the level surfaces of r) by constructing all surfaces which, taken together with S, intercept equal lengths of the congruence of space curves formed by the intersection of the level surfaces of r and φ. Example: cylindrical coordinates with x-axis as axis of symmetry; here r is distance from the axis, φ is an angle between half planes bounded by the axis, and S is, say, the (y, z)-plane. The equations of motion governing the flow in a turbine rotating with uniform angular speed ω about the x-axis are

given in terms of the coordinates r, φ, s, and the possible forms of the turbine blades are determined for two flows satisfying certain specified conditions. *J. B. Diaz.*

Stanitz, John D. Two-dimensional compressible flow in conical mixed-flow compressors. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1744**, 75 pp. (1948).

By a conical mixed flow compressor the author means one for which the average flow curve is a line, generating a right circular cone K about the axis of the impeller. Let α denote the cone angle of K and (r, θ) polar coordinates on K, the angle θ being measured relative to a direction fixed in the impeller. It is assumed that the passage height h is a function of r alone and that all flow properties within the passage are functions of r and θ only. Let the values of r and h at the blade tip be denoted by r_T and h_T and the tangential and radial components of the velocity relative to the impeller be denoted by u and v, respectively. Then the equation of continuity takes a form which permits the definition of a dimensionless stream function $\psi(R, \theta)$ by the equations $\rho_0 c_0 \psi_\theta = \rho v H R$, $\rho_0 c_0 \psi_R = -\rho u H$, where $R = r/r_T$ and $H = h/h_T$, and where ρ is the density and ρ_0 and c_0 are the density and local sound velocity at absolute inlet stagnation conditions. The requirement that the absolute motion be irrotational leads then to a nonlinear partial differential equation for ψ,

$$\psi_{RR} + (\psi_R/R) + (\psi_{\theta\theta}/R^2) - (\log H)_R \psi_R - [\log(\rho/\rho_0)]_R \psi_R - [\log(\rho/\rho_0)]_\theta (\psi_\theta/R^2) = 2M_T(\rho/\rho_0)H,$$

where M_T is the Mach number of the impeller tip. By use of the energy equation, a relation is obtained for the density ratio ρ/ρ_0 in terms of the speed $q = (u^2+v^2)^{\frac{1}{2}}$, the isentropic exponent and a temperature correction for the whirl ahead of the impeller. In place of R and θ, the author then introduces as independent variables ξ and η, which are respectively the velocity potential and stream function of the corresponding incompressible flow. The corresponding analytic function $f(z) = f(Re^{i\theta}) = \xi - i\eta$ maps the blades into the straight lines η = constant. Upon a lattice constructed between two such successive lines, the author applies the relaxation methods of R. V. Southwell as elaborated by H. W. Emmons [Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **932** (1944); these Rev. **7**, 226]. Further details, analytical and graphical, are given in the special cases that the trace of each blade on the cone K is a logarithmic spiral or a ray θ = constant and that the passage height is given by the law $H = R^m$. *M. Marden* (Milwaukee, Wis.).

Roy, Maurice. Couche-limite laminaire sur l'hélicoïde en viration uniforme. C. R. Acad. Sci. Paris **228**, 1994–1996 (1949).

The author considers the uniform laminar flow of a fluid with small kinematic viscosity ν over a rotating semi-infinite helicoid of one sheet. The helix angle, the free stream velocity V and the angular velocity ω of the helicoid are related so that the fluid meets the leading edge of the helicoid tangentially. The problem is essentially an idealization of the boundary layer flow on the blade of an axial turbomachine. By writing the Navier-Stokes equations in curvilinear coordinates s, n, r, where s is the distance along the surface from the leading edge, n the distance normal to the surface, and r the radius from the axis of revolution, and by deleting small quantities, the author arrives at the logical generalization of the boundary layer equations valid for the rotating helicoid. These are

$$\frac{\partial v_s}{\partial s} + \frac{\partial v_n}{\partial n} + \frac{\partial v_r}{\partial r} + \frac{v_r}{r} = 0,$$

$$v_s\frac{\partial v_s}{\partial s} + v_n\frac{\partial v_s}{\partial n} + v_r\frac{\partial v_s}{\partial r} + \frac{v_r \sin\beta}{r}(v_s \sin\beta - 2\omega r) = \nu\frac{\partial^2 v_s}{\partial n^2},$$

$$v_s\frac{\partial v_r}{\partial s} + v_n\frac{\partial v_r}{\partial n} + v_r\frac{\partial v_r}{\partial r} - \frac{(v_s \sin\beta - \omega r)^2}{r} = \nu\frac{\partial^2 v_r}{\partial n^2},$$

where v_s, v_n, v_r are the local velocity components and $\tan\beta = \omega r/V$. In particular, the author finds that the pressure gradient normal to the surface of the helicoid cannot be neglected. The author limits his discussion to results which may be observed directly from the equations. It is pointed out that, near the surface, the gradient of the radial velocity may, because of centrifugal force, exceed the gradient of the velocity in the direction of flow. Also, because of the radial boundary layer flow, a trend toward separation at large distances from the leading edge is exhibited.

F. E. Marble (Pasadena, Calif.).

Loĭcyanskiĭ, L. G. The resistance of a lattice of profiles in a gas flow with sub-critical velocity. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 171–186 (1949). (Russian)

A method for defining and calculating the resistance of a lattice in a compressible viscous fluid is given. The assumption is made that the local Mach number is everywhere less than unity. The case of incompressible flow, previously treated by the same author [same journal **11**, 449–458 (1947); these Rev. **9**, 163] appears as a special case of the present theory. Due to the assumption of subsonic flow, a formula for the lifting force similar to that of Joukowski may be used. This makes possible a rigorous definition of the term resistance of a profile in a lattice. An expression for the resistance of a single profile may be obtained by the well-known method of applying the momentum theorem. In the case of a lattice, a difficulty arises from the fact that the wakes from the different profiles join at a finite distance downstream. The concept of an external potential flow thus loses its meaning and a treatment of the wake with methods from boundary layer theory is not possible. This difficulty is resolved by considering the relations between the flow variables at the plane where the wakes join and the corresponding quantities at a plane infinitely far downstream. Here the essential assumption is made that this shear flow differs only slightly from uniform flow so that a linearized treatment is possible. A similar assumption was made in the paper quoted above and subsequently shown to be in good agreement with experiment. The author's method is independent of what particular method for treating the compressible boundary layer is used. *P. A. Lagerstrom.*

Loĭcyanskiĭ, L. G. A generalization of Joukowski's formula for profiles in a lattice in a gas flow at subsonic velocity. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 209–216 (1949). (Russian)

Joukowski's formula for the lifting force on a two-dimensional wing in an ideal incompressible fluid is generalized to the case of compressible flow past a lattice of profiles. It is shown that if the local Mach number is everywhere less than unity the ordinary form of Joukowski's formula is an approximation sufficient for practical purposes. The arithmetic mean of the velocity vectors infinitely far up-

stream and downstream should be used as "free stream velocity." *P. A. Lagerstrom* (Pasadena, Calif.).

Loytsiansky, L. G. Reaction of the boundary layer on the distribution of pressure over the surface of a body in a flow of real fluid. Tech. Rep. no. **F-TS-1210-IA (GDAM A9-T-28)**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. ii+21 pp. (1949).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 205–214 (1947); these Rev. **9**, 543.

Hartree, D. R. A solution of the laminar boundary-layer equation for retarded flow. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2426 (3963)**, 27 pp. (1949).

Numerical results.

Hartree, D. R. The solution of the equations of the laminar boundary layer for Schubauer's observed pressure distribution for an elliptic cylinder. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2427 (3966)**, 30 pp. (1949).

Numerical results.

Young, A. D. Skin friction in the laminar boundary layer in compressible flow. Aeronaut. Quart. **1**, 137–164 (1949).

This paper is essentially the same as the author's previous paper by the same title [Coll. Aeronaut. Cranfield. Rep. no. **20** (1948); these Rev. **10**, 755]. *F. E. Marble.*

Lees, Lester. The stability of the laminar boundary layer in a compressible fluid. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **876**, i+47 pp. (1947).

Formerly issued as Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1360** (1947); these Rev. **9**, 116.

Chapman, Dean R., and Rubesin, Morris W. Temperature and velocity profiles in the compressible laminar boundary layer with arbitrary distribution of surface temperature. J. Aeronaut. Sci. **16**, 547–565 (1949).

An analysis is presented which enables the temperature profile, velocity profile, heat transfer, and skin friction to be calculated for laminar boundary layers on a two-dimensional or axially symmetric surface without pressure gradient but with an arbitrary analytic distribution of surface temperature. In addition to constant pressure, a constant specific heat and Prandtl number are assumed, and viscosity is set proportional to absolute temperature. On this basis and by the use of a suitable velocity potential, the equation of motion is reduced to the Blasius form. With the assumption of viscosity variation and constant pressure, the equation of motion is independent of energy so that an integration is possible. The temperature profile corresponding to an arbitrary temperature distribution is obtained by expansion in a set of profile functions with coefficients from the Taylor expansion of the temperature distribution. Results for skin friction and heat transfer are compared with solutions based on other assumptions as to viscosity variation and are shown to be in close agreement. The detailed form of the solution is examined for a specific temperature distribution at both subsonic and supersonic velocities. *N. A. Hall* (Minneapolis, Minn.).

Švec, M. E. On the approximate solution of some problems of the hydrodynamics of the boundary layer. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 257–266 (1949). (Russian)

The gradients occurring in a laminar boundary layer are obtained by a method of successive approximation which consists essentially of assuming initially a linear gradient satisfying the boundary conditions and adding successive corrections obtained by converting the exact boundary layer partial differential equation to a recurrence equation. The method is applied to a diffusion boundary layer, to the normal velocity boundary layer, to the thermal boundary layer, and an approach is indicated for turbulent boundary layers. The results give the gradients as polynomials in the thickness parameter $\alpha^2 x^{-1}$ as with the well-known Pohlhausen solution. Comparison with exact solutions indicate good accuracy for the second order approximation. *N. A. Hall* (Minneapolis, Minn.).

Thwaites, B. Approximate calculation of the laminar boundary layer. Aeronaut. Quart. **1**, 245–280 (1949).

After analyzing a large class of boundary-layer velocity-profiles, the author discovered that the functions $l(m)$ and $H(m)$ for all such cases differ only slightly from each other over the whole range of positive and negative pressure gradients. Here l, m and H are defined by $(\partial u/\partial y)_{y=0}=(U/\vartheta)l$, $(\partial^2 u/\partial y^2)_{y=0}=(U/\vartheta^2)m$ and $H=\delta^*/\vartheta$, u being the velocity-component in the x direction and U the value of u at the edge of the boundary-layer and δ^* and ϑ the displacement and momentum thickness, respectively. Based on this discovery, an approximate method is proposed by constructing two universal curves $l(m)$ and $H(m)$ for all conceivable boundary-layer flows found in practice. Once these are chosen, Kármán's momentum-integral can be written in the form $(U/2\nu)d\vartheta^2/dx=M(\vartheta^2, U)$, ν being the kinematic viscosity coefficient, and can be integrated numerically. As examples, both Howarth's $(U=\beta_0-\beta_1 x)$ and Hartree's (ellipse) exact solutions are checked. The agreement obtained is considered good for practical purposes.

Y. H. Kuo (Ithaca, N. Y.).

Yuan, Shao Wen. Heat transfer in laminar compressible boundary layer on a porous flat plate with fluid injection. J. Aeronaut. Sci. **16**, 741–748 (1949).

To investigate the problem of cooling of combustion chambers by injection, the flow of a hot gas over a porous flat plate is considered under the assumption that (1) the density of the gas is inversely proportional to the temperature and the viscosity is proportional to the square root and three-fourths power of temperature; (2) the Prandtl number is unity and (3) the flow is laminar and the injection is uniform along the plate. By assuming the velocity profile as a polynomial of the fourth degree, satisfying a set of conditions, the Kármán momentum-integral is integrated to give the shape of the boundary-layer. It is shown that the boundary-layer thickness is almost a linear function of the distance from the leading edge, increasing as the temperature ratio between the edge of the boundary-layer and the plate is raised. The ratio of the temperature difference between the edge of the boundary-layer and the plate to that between the plate and the coolant is evaluated in terms of mass injection. For constant value of this ratio, the rate of injection was found to increase as the Mach number increases and to decrease as the Reynolds number is raised. In one case, the result is compared with experiment with a measure of agreement. *Y. H. Kuo* (Ithaca, N. Y.).

Chiarulli, P. Stability of two-dimensional velocity distributions of the half-jet type. Prepared under the supervision of C. C. Lin. Tech. Rep. no. **F-TS-1228-IA (GDAM A9-M-VI(Sup. 1))**. Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. iv+51 pp. (1949).

The stability of a plane jet of incompressible fluid is studied, using Lin's ideas [Quart. Appl. Math. **3**, 117–142, 218–234 (1945); 277–301 (1946); these Rev. **7**, 225, 226, 346]. However, critical Reynolds numbers are not computed numerically. *M. J. Lighthill* (Manchester).

Pai, S. I. Two-dimensional jet mixing of a compressible fluid. J. Aeronaut. Sci. **16**, 463–469 (1949).

The author considers both the laminar and turbulent constant pressure mixing of a compressible fluid of density ρ and viscosity μ at a plane velocity discontinuity and in a two-dimensional jet parallel to the x-axis. For laminar motion with velocity u in the x-direction and small velocity v in the y-direction, the equations of continuity, motion, and energy may be simplified to

$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=0,$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right),$$

$$\left(\rho u\frac{\partial}{\partial x}+\rho v\frac{\partial}{\partial y}\right)C_P T=\frac{\partial}{\partial y}\left(\lambda\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2,$$

where T is the fluid temperature and C_P and λ are the specific heat at constant pressure and the thermal conductivity, respectively. The author obtains his solutions by employing the iteration procedure developed by von Kármán and Tsien [J. Aeronaut. Sci. **5**, 227–232 (1938)] in connection with their solution for the boundary layer on a flat plate. The initial approximation for the iteration procedure was obtained by assuming all differences of velocity to be small and using a perturbation method. This problem for u_1, the first approximation to the solution, then resolves to one of the heat conduction type

$$\frac{\partial u_1}{\partial x}=\frac{\mu_0}{\rho_0 u_0}\frac{\partial^2 u_1}{\partial y^2},$$

explicit solutions of which are known. The parameters ρ_0, μ_0, and u_0 correspond to the mean conditions about which the perturbation was taken. Treatment of velocity discontinuity and jet problems differs only in the boundary conditions. The results show that the velocity gradients are decreased as the jet temperature is increased but the rate of spread is only slightly increased. The increase of laminar diffusion rate, due to the increased viscosity and conductivity of the heated fluid, is more marked for the mixing of two streams than for the jet.

Analysis of the turbulent mixing process is based on the assumption of a turbulent mixing stress τ of the form suggested by Reichardt [Z. Angew. Math. Mech. **21**, 257–264 (1945)]:

$$\tau=\rho k(u_{\max}-u_{\min})b\partial u/\partial y,$$

where b is the breadth of the mixing zone, k is an experimentally determined constant, and $u_{\max}$ and $u_{\min}$ are the maximum and minimum velocities occurring in the mixing process at a given value of x. The author further assumes, with Frankl [Tech. Memos. Nat. Adv. Comm. Aeronaut., no. **1032**, 1–8 (1942); these Rev. **4**, 121] that the temperature-velocity integral $T=A+Bu-\frac{1}{2}u^2/C_P$, valid for laminar motion with Prandtl number $C_P\mu/\lambda=1$, holds also for the turbulent flow. Under these restrictions the problems of the turbulent mixing of two parallel streams and mixing of a jet are reduced, through a simple transformation, to essentially the same mathematical problem as the laminar diffusion. The Kármán-Tsien procedure may then be applied.

F. E. Marble (Pasadena, Calif.).

Burgers, J. M. Note on the damping of the rotational oscillation of a spherical mass of an elastic fluid in consequence of slipping along the boundary. Nederl. Akad. Wetensch., Proc. **52**, 111–119 (1949). (Esperanto summary)

Further approximations are given to the damped oscillation of a sphere of fluid possessing elastic and viscous properties, which was previously treated by the author to a first approximation in the rate of decay [same Proc. **51**, 1211–1221 (1948); these Rev. **10**, 646]. *M. J. Lighthill.*

Thomas, T. Y. On conditions for steady plane flow with shock waves. J. Math. Physics **28**, 91–98 (1949).

In steady two-dimensional supersonic flow past a pointed symmetric obstacle there are consistency relations which hold along an attached shock [J. Math. Physics **28**, 62–90 (1949); these Rev. **10**, 758]. These take the form

$$d^n K/d\eta^n=G_n(M,\alpha)d^n k/ds^n+H_n,$$

where K and k are the curvatures of streamline and shock at the shock, η and s are arc lengths along streamline and shock, and the G_n are rational functions of M^2 (free flow Mach number) with coefficients depending on α, the shock inclination angle. In what follows these relations are taken at the vertex of the obstacle. A value of α for which any G_n vanishes for fixed $M>1$ is called "singular relative to M." The author considers a time dependent shock line, with unsteady flow behind it, uniform flow in front. He then shows that a sufficient condition that the shock and the flow behind it be steady is that α be nonsingular relative to M.

D. P. Ling (Murray Hill, N. J.).

Thomas, T. Y. The fundamental hydrodynamical equations and shock conditions for gases. Math. Mag. **22**, 169–189 (1949).

In this expository paper the author develops in a clear and rapidly paced fashion the fundamental mathematical apparatus appropriate to the science of gas dynamics, and formulates side by side with these the physical principles on which this science rests. The physical laws are translated into integral form in the first instance. Time differentiation of integrals over moving volumes is explored, and the laws are then finally cast in their differential form. The distortion and viscosity tensors are introduced and related, leading to the Navier-Stokes equations. Also a brief discussion of the thermodynamics of ideal gases allows the formulation of the heat equation. Finally, the mathematical machinery previously introduced is used to derive the shock conditions for ideal gases. The paper is aimed at students possessing some mathematical and general scientific maturity. A familiarity with Cartesian tensors is assumed, for example, on the mathematical side, and on the physical side the reader is expected to be acquainted with the reduction of the stress-strain relations to their canonical form.

D. P. Ling (Murray Hill, N. J.).

Cabannes, Henri. Détermination approchée de l'onde de choc détachée. C. R. Acad. Sci. Paris **229**, 492–493 (1949).

The steady supersonic axi-symmetrical flow of a uniform stream of air past a blunt-nosed body of revolution is investigated, neglecting viscosity and thermal conductivity, and using the Crocco stream function suitable for the problem. Assuming that this function is adequately approximated by three terms of a Taylor expression in the region directly ahead of the nose, the author finds the position and radius of curvature of the shock in terms of the Mach number and the radius of curvature of the nose.

M. J. Lighthill (Manchester).

John, Fritz. On the motion of floating bodies. I. Comm. Pure Appl. Math. **2**, 13–57 (1949).

This paper is divided into three parts. The first part derives the fundamental equations and boundary conditions for the complete nonlinear theory of motion for a heavy fluid with free surface, movable solid boundaries (floating bodies) and fixed boundaries. In the second part the linearized theory is derived by means of expansions of the fundamental functions in powers of a parameter. From the linearized equations are derived the conditions for equilibrium and stability with respect to small disturbance of a floating body. The derivation of the classical sufficient condition for stability has an advantage over the usual procedure in that hydrodynamic forces arising from motion of the body are not neglected ab initio. A uniqueness theorem is also established for the subsequent motion of fluid and body for the case of an initial small disturbance. The third part is devoted to motion of a floating body in a fluid of finite depth. The shallow water theory (wave length and radius of curvature of body both large compared with depth) is derived by further expansions in powers of another parameter, following a method used previously by K. O. Friedrichs. This is further specialized to the case of two dimensions and the following problems are discussed: (A) waves generated by a freely floating cylinder; (B) waves generated by a forced vertical motion of a cylinder; (C) incoming waves disturbed by a rigid cylinder; (D) incoming waves and a floating cylinder; (E) vertical motion of a floating body of revolution. In the first four cases some numerical computations have been made and graphs of some pertinent quantities presented. *J. V. Wehausen.*

John, Fritz. The action of floating bodies on ocean waves. Ann. New York Acad. Sci. **51**, 351–359 (1949).

This is an expository lecture describing the results in the paper reviewed above as well as some of those that presumably will be in part II of that paper, as yet unpublished. These additional results treat the case of motions in a liquid of infinite depth which are periodic and simple harmonic functions of the time. From the brief description there appears to be some duplication of work by Haskind [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Meh.] **10**, 33–66 (1946); these Rev. **11**, 228]. *J. V. Wehausen.*

Ursell, F. On the rolling motion of cylinders in the surface of a fluid. Quart. J. Mech. Appl. Math. **2**, 335–353 (1949).

The author investigates the rolling motion of a long floating cylinder when the period is long and the amplitude small. For this case an approximate value for the stream function and for the amplitude at infinity is derived. A second approach is made to the problem of finding an approximate expression for the waves at infinity by starting from the solution obtained when the free surface is replaced by a rigid plane, a method which is also applied to the case of heaving motion. The results are applied to a cylinder of elliptical section to find the roll axis in slow rolling, the added mass for forced periodic heaving motion of small amplitude and long period, and the motion in long waves. Finally, the damping of rolling caused by the generated waves is discussed for sections similar to ship sections (squares with rounded corners) and an approximate expression derived for the amplitude at infinity of these waves. For one member of the family discussed the amplitude vanishes to the order of approximation considered so that little wave damping of the roll may be expected. There are apparently no experimental data to compare with.

J. V. Wehausen (Falls Church, Va.).

Lowell, Sherman Cabot. The propagation of waves in shallow water. Comm. Pure Appl. Math. **2**, 275–291 (1949).

The propagation of waves of small amplitude in shallow water is determined by a linear partial differential equation of hyperbolic type in variables x, y, t. The author first transforms this differential equation into an integral equation by Green's theorem and then derives equations for the propagation of discontinuities in the surface elevation, imposing the condition that discontinuous solutions have to be the limiting cases of continuous solutions. One of these equations is the equation of the wave fronts which is identical to the eiconal equation in the geometrical optics. The second equation gives the variation of the magnitude of the discontinuities along the rays which are orthogonal to the wave fronts. Methods for constructing the wave fronts and rays are given. The author then shows that, for wave motions periodic in time, the previous results for the propagation of discontinuities can be used if the wave length is very small. The last section of the paper treats the problem of oblique incidence of waves on a uniformly sloping beach.

H. S. Tsien (Pasadena, Calif.).

Høiland, Einar. Stability and instability waves in sliding layers with internal static stability. Arch. Math. Naturvid. **50**, no. 3, 71–94 (1949).

Let an incompressible fluid be subject to a gravitational force, be bounded by two horizontal planes, and have a density varying according to the barometric law $\rho=\rho_0 \exp(-\Gamma g_z)$. The author considers the stability of the two-dimensional motion occurring when the velocity of the fluid has a discontinuity at a horizontal plane but is constant on either side. Linearized equations for the perturbed motion and harmonic perturbations are assumed. The paper is devoted chiefly to a general discussion of the properties of the equation relating the velocity of propagation of the disturbance and its wave length. *J. V. Wehausen.*

Rubinšteĭn, L. I. On a hydrodynamical problem. Doklady Akad. Nauk SSSR (N.S.) **68**, 821–824 (1949). (Russian)

L'auteur cherche à construire un écoulement permanent, plan et lent d'un fluide visqueux incompressible, s'effectuant dans le domaine D: $0\leqq x\leqq a$; $0\leqq z\leqq c$, du plan Oxz, sous l'action de la pesanteur, des efforts tangentiels constants le long de la surface libre L: $z=\zeta(x)$ (a priori inconnue), et des forces de Coriolis, dues à la rotation de la terre, calculées en négligeant la composante horizontale de la rotation terrestre. On admet de plus que L diffère peu de $z=0$, en sorte que les conditions frontières seront considérées comme

satisfaites, si elles le sont pour $z=0$. L'auteur énonce le problème aux limites auquel se ramène la question et en obtient la solution sous forme de développements en série dont il établit la convergence et l'unicité.

J. Kravtchenko (Grenoble).

Yaglom, A. M. On the acceleration field in a turbulent flow. Doklady Akad. Nauk SSSR (N.S.) **67**, 795–798 (1949). (Russian)

A general theory of locally isotropic turbulence at high Reynolds numbers has been developed by Kolmogoroff [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 301–305 (1941); **32**, 16–18 (1941); these Rev. **2**, 327; **3**, 221] and Obuhov [ibid., 19–21 (1941); Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1941**, 453–466; same Doklady (N.S.) **66**, 17–20 (1949); **67**, 643–646 (1949); these Rev. **3**, 221; **4**, 121; **10**, 757; **11**, 66; see also the following review]. In the present paper the author applies this theory to the study of the acceleration field in a turbulent flow by considering the mean square of the acceleration at a point of a fluid as the characteristic quantity. From the Navier-Stokes differential equations it follows that

$$\sum_{i=1}^{3}\overline{(dv_i/dt)^2}=\rho^{-2}\sum_{i=1}^{3}\overline{(\partial p/\partial x_i)^2}+\nu^2\sum_{i=1}^{3}\overline{(\Delta v_i)^2}-(2\nu/\rho)\sum_{i=1}^{3}\overline{(\partial p/\partial x_i)\Delta v_i}.$$

Neglecting the third term on the right and expressing the first two terms by means of the structure functions of the field of pressure and the field of velocities, respectively, the author, following Obuhov [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 19–21 (1941); these Rev. **3**, 221], Millionščikov [Bull. Acad. Sci. URSS. Sér. Géograph. Géophys. [Izvestia Akad. Nauk SSSR] **1941**, 433–446; these Rev. **4**, 121] and Kolmogoroff, gets the following expressions:

$$\rho^{-2}\sum_{i=1}^{3}(\partial p/\partial x_i)^2=-1.1s_0^{-1}\nu^{-1}|\epsilon|^3,\quad \nu^2\sum_{i=1}^{3}(\Delta v_i)^2=-0.3s_0\nu^{-1}|\epsilon|^3,$$

where $s_0=-0.4$ [cf. Townsend, Proc. Cambridge Philos. Soc. **44**, 560–565 (1948); Batchelor, ibid. **43**, 533–559 (1947); these Rev. **9**, 97]. Comparison of the last two expressions shows that the acceleration of particles in a turbulent flow at high Reynolds numbers is dominated by the fluctuations of pressure gradients [cf. Heisenberg, Z. Physik **124**, 628–657 (1948); these Rev. **11**, 63]. *E. Leimanis.*

Obuhov, A. M. Concerning the geostrophic wind. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **13**, 281–306 (1949). (Russian)

L'auteur étudie la déviation du vent réel par rapport au vent géostrophique. En partant des équations du mouvement dans le champ de la force de Coriolis, l'auteur néglige les termes non linéaires ainsi que la composante verticale du vent. En prenant comme nouvelles variables les moyennes suivant l'altitude des composantes de la vitesse horizontale et de la pression et en supposant le fluide barotrope, l'auteur obtient un système d'équations aux dérivées partielles qui caractérise le mouvement d'une pellicule liquide compressible à deux dimensions immergée dans un champ de force de Coriolis. Pour résoudre le système, on décompose les composantes de la vitesse moyenne suivant l'altitude en deux parties dont l'une est une fonction du courant et l'autre un potentiel des vitesses. L'auteur résoud le système pour un champ de vitesse stationnaire; il obtient un résultat analogue à celui de Jeffreys. Pour le cas non stationnaire et en négligeant tous les termes quadratiques on obtient l'invariance par rapport au temps d'une certaine fonction que l'auteur désigne sous le nom "tourbillon potentiel" et qui est une conséquence du théorème de Helmholtz. En égalant le tourbillon potentiel à zéro on obtient un mouvement ondulatoire que l'auteur étudie en détail. Les ondes se propagent avec une vitesse voisine à celle du son, donc se présentent rarement dans l'atmosphère.

On peut donc considérer que la solution générale du problème linéaire peut être représentée comme une somme de deux composantes: l'une correspondante à un champ stationnaire et l'autre à un champ ondulatoire. L'auteur étudie la solution du système linéaire et explique l'adaptation du champ hydrodynamique de l'atmosphère. Supposons qu'à l'instant initial le vent coïncide partout avec le vent géostrophique, sauf dans un domaine limité représenté par un cercle de rayon R; cette anomalie dans la distribution du vent provoque la formation des ondes qui se propagent avec la vitesse du son. Avec le temps l'énergie de la composante ondulatoire se dissipe et le mouvement tend vers la composante stationnaire, ce qui représente le processus d'adaptation du champ. C'est un processus assez rapide, par exemple pour un rayon de 500 km, le temps d'adaptation est de l'ordre d'une heure.

L'auteur donne un exemple concret pour illustrer sa théorie. Il trouve que la valeur géostrophique de la pression au centre du système, pour les conditions choisies, s'établit environ dans un temps de 3 à 4 heures. Le mémoire se termine par une étude du système non linéaire. Pour étudier ce cas compliqué l'auteur utilise les développements suivant un très petit paramètre. *M. Kiveliovitch* (Paris).

Kiveliovitch, M. Sur la composante verticale du vent. J. Sci. Météorologie **1**, 18–22 (1949). (French. English and Spanish summaries)

An equation is derived for the vertical wind velocity under the assumption of geostrophic wind, adiabatic changes, and a lapse rate of temperature constant in space. This equation relates the vertical velocity either to the pressure change at the same level or to surface temperature and pressure changes. Due to the assumption of geostrophic wind, the important term containing horizontal divergence drops out. *H. Panofsky* (New York, N. Y.).

Kiveliovitch, M. Note sur une méthode de Kibel pour la prévision du temps. J. Sci. Météorologie **1**, 72–74 (1949). (French. English and Spanish summaries)

This note criticizes a method for the numerical integration of the meteorological equations proposed by Kibel. Kibel's equations postulated no vertical motion, geostrophic wind and adiabatic temperature changes. The present note shows that, if the unrealistic assumption is made of a lapse rate of temperature constant in space, Kibel's equation leads to the paradoxical result that pressure and temperature do not change locally. *H. Panofsky* (New York, N. Y.).

Van Mieghem, J. Production et redistribution de la quantité de mouvement et de l'énergie cinétique dans l'atmosphère. Application à la circulation atmosphérique générale. J. Sci. Météorologie **1**, 53–67 (1949). (French. English and Spanish summaries)

The author rederives equations expressing the balance of momentum, kinetic energy, potential energy, internal energy and entropy. It is shown that kinetic energy is produced in regions of divergence and destroyed in regions of convergence. The tropical and polar regions both are found to

be sources of the momentum and kinetic energy in the westerlies in mid-latitudes. *H. Panofsky.*

Linelkin, P. S. Theory of convection in a rectangular basin. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. **13**, 393–408 (1949). (Russian)

This is a contribution to a mathematical theory which might explain the monsoonal air currents from oceans to continents. The model of the problem considered is a rectangular basin filled with a liquid of uniform density. The walls and the top of the basin are insulated against heat escape. The bottom consists of two equal rectangular plates made of different materials. One of these plates is heated, while the other is kept at a constant temperature. The circulation currents within the liquid which arise during the experiment are to simulate the processes of atmospheric convection peculiar to a monsoon. The theory treats the case after the steady heat flow has set in. The velocity potential of the liquid is found as the solution within a rectangular region of a nonhomogeneous biharmonic equation. The solution is found by the method of successive approximations. Numerical computations are made, and the lines of flow are plotted for a particular case. The maximum and minimum horizontal velocity components are found and located for this particular case.

H. P. Thielman (Ames, Iowa).

Hidaka, Koji. Mass transport in ocean currents and lateral mixing. J. Marine Research **8**, 132–136 (1949).

***Strecke, Joachim. Mathematische Innenballistik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 7, pp. 187–209. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

This is a review of progress by German ballisticians during the indicated period in the applied theory of interior ballistics. The earlier fundamental theory of Résal, Sarrau and Vieille provides adequate answers to the more immediate problems, and amply justifies predictions concerning maximum powder gas pressures and muzzle energies. The influence of the viscosity of combustion gas and of its eddying upon the temperature loss is not known. The theory must therefore as yet be content with idealized reactions and compute with linear gas flow. Recent investigations provided at least opportunity to formulate laws concerning the distribution of heat loss at the bore surface. The author reviews the dimension-free theory of Résal. Charbonnier's assumption is simplified for the modern powder with almost constant burning surface. The general linearized equations with certain further simplifying assumptions, such as assuming the covolume of the combustion gas to be equal to the specific volume of the fixed power, may be integrated in finite form to give the muzzle energy and maximum gas pressure. Theoretical results and observations not yet explained on theoretical grounds are discussed, particularly with regard to effect of the density of loading upon muzzle velocity. Theoretical and experimental results concerning the pressure difference upon the projectile base and the rear of the powder chamber due to flow of the gas and powder are given. Various theories as to the maximum possible muzzle velocity are discussed. The inertia of the air in the bore in front of the projectile has a noticeable effect. *A. A. Bennett* (Providence, R. I.).

Brillouin, Jacques. Démonstration directe des formules de Fresnel pour la diffraction d'une onde plane par un demi-plan réfléchissant. C. R. Acad. Sci. Paris **229**, 513–514 (1949).

The diffraction of a monochromatic, plane wave of sound by a perfectly reflecting half plane is treated by the introduction of parabolic coordinates. [For a rather complete bibliography on this problem, see H. Bateman, The Mathematical Analysis of Electrical and Optical Wave Motion on the Basis of Maxwell's Equation, Cambridge University Press, 1915, pp. 82–88.] The results are used to establish directly Fresnel's formulae for this particular problem.

J. W. Miles (Los Angeles, Calif.).

Billing, H. Geradlinig bewegte Schallquellen. Z. Angew. Math. Mech. **29**, 267–274 (1949). (German. Russian summary)

The velocity potential of a source moving with subsonic velocity and having a strength which exhibits a harmonic time dependence is calculated from the linearized equations using an attack due essentially to H. G. Küssner [Luftfahrtforschung **17**, 370–378 (1940); these Rev. **2**, 330]. The results are applied to a study of the pressure field of a circular cylinder due to the harmonically varying circulation associated with the formation of the Kármán vortex street. The disagreement between the author's results and previously known experimental measurements is attributed to the presence of strong turbulence in the incident flow in the latter case. *J. W. Miles* (Los Angeles, Calif.).

Stenzel, H. Über die Berechnung des Schallfeldes von kreisförmigen Membranen in starrer Wand. Ann. Physik (6) **4**, 303–324 (1949).

In order to determine the sound pressure on a circular membrane in a rigid wall at a point with coordinates (r, γ) two auxiliary functions are introduced (r is the distance from a point in space to the center of the circle, and γ is the angle which this radius makes with the perpendicular passing through the center of the circle). Some applications are given. *A. E. Heins* (Pittsburgh, Pa.).

Leitner, Alfred. Diffraction of sound by a circular disk. J. Acoust. Soc. Amer. **21**, 331–334 (1949).

Kirchhoff's approximation and recent experimental data are compared with the exact theory of diffraction of sound by a circular disk. *A. E. Heins* (Pittsburgh, Pa.).

Mawardi, Osman K. Generalized solutions of Webster's horn theory. J. Acoust. Soc. Amer. **21**, 323–330 (1949).

Webster's equation [Proc. Nat. Acad. Sci. U. S. A. **5**, 275–282 (1919)] for the approximate formulation of the propagation of sound waves in horns is treated by two methods. A transmission line with variable parameters is considered as the electrical analogue of the horn and yields information regarding horns of finite length. The second method studies the singular solutions of the Webster differential equation. *A. E. Heins* (Pittsburgh, Pa.).

Elasticity, Plasticity

Clark, G. L. The problem of a rotating incompressible disk. Proc. Cambridge Philos. Soc. **45**, 405–410 (1949).

This paper continues the author's previous work on the mechanics of continuous matter [Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 434–441 (1949); these Rev. **11**, 59].

Taking the stress tensor to be diagonal with elements $(p, 0, 0)$, $(p, p, 0)$, or (p, p, p), he assumes as stress-strain law for so-called incompressible matter (*) $p=\alpha c^2\Delta(1+\Delta)^{-1}$, where α is a constant (unstrained density), c the velocity of light, and Δ the dilatation. On the basis of equations derived from an energy tensor, he concludes that plane waves of dilatation are propagated with velocity c in a medium satisfying (*). He then investigates the external and internal fields of a rotating disk, and draws the conclusion that a disk formed of material satisfying (*) undergoes no change in radius when made to rotate. *J. L. Synge* (Dublin).

Fadle, J. Eine einfache vektoranalytische Ableitung der Grundgleichungen der Elastomechanik für orthogonale, krummlinige Koordinaten. Ing.-Arch. **17**, 62–70 (1949).

Rachkovitch, D. Les équations fondamentales de la théorie d'élasticité (en interprétation géométrique) sous forme vectorielle. Godisnjak Tehn. Fak. Univ. Beograd. **1946–47**, 9–12 (1949). (Serbian. French summary)

Aržanyh, I. S. Integral equations of the theory of elasticity. Uspehi Matem. Nauk (N.S.) **4**, no. 5(33), 176–177 (1949). (Russian)

Under consideration are the surface and volume vector potentials:

$$\bar{v}(q)=\int\int_S L(q, p)\bar{\sigma}(p)dS_p, \qquad \bar{U}(q)=\int\int\int_T L(q, p)\bar{\tau}(p)dT_p,$$

$$\bar{w}(q)=\int\int_S K(q, p)\bar{\rho}(p)dS_p, \qquad \bar{E}_u(q)=\int\int\int_T \bar{D}(\bar{u})dT_p;$$

L, K are certain tensors; the vector $\bar{D}(\bar{u})$ is determined by the component tensors of deformation. It is assumed that the closed surface S satisfies the Liapounoff conditions of "smoothness." It is indicated that certain systems of integral equations are to be solved for each of the four classical problems of elasticity (the first and second, interior and exterior problems). The solution of any of these systems enables determination of the corresponding vector of deformation. *W. J. Trjitzinsky* (Urbana, Ill.).

Kondo, Kazuo. On the statics of membrane. J. Soc. Appl. Mech. Japan **2**, 1–2 (1949). (Japanese)

The problems of a two-dimensional membrane covering the smooth surface of a curved body are extended to an arbitrary number of dimensions and the concept of a constraining manifold is introduced. The statics of such a state are connected with the general geometrical properties of a subspace in a Riemannian space. Two principal results have been obtained as follows: (1) the condition of equilibrium is, here also, the vanishing of the divergence of the stress tensor in connection with the constraining manifold; (2) the constraining force is given by an expression containing the Euler-Schouten curvature tensor. *A. Kawaguchi* (Sapporo).

Kondo, Kazuo. The geometry of the perfect tension field. I. J. Soc. Appl. Mech. Japan **2**, 3–4 (1949). (Japanese)

The definitions and the basic concepts of a perfect tension field, due to H. Wagner, are generalized so as to include, first the case of three-dimensional Euclidean space and, next, a Riemannian space of any number of dimensions (although the practical cases appearing in the theory of light structures and of membranes are not so general). The two fundamental theorems due to Wagner are so modified as to suit the new concepts. They assert that: (1) the divergence of the stress vector (not of the tensor) vanishes in the tension field; (2) any trajectory of the stress vector is a geodesic of the constraining Riemannian space. These theorems hold also for the special cases: (i) of a membrane constrained in a definite curved surface (a constrained system), (ii) of a free membrane in three-dimensional Euclidean space (a free system). *A. Kawaguchi.*

Kondo, Kazuo. A proposal of a new theory concerning the yielding of materials based on Riemannian geometry. J. Soc. Appl. Mech. Japan **2**, 29–31 (1949). (Japanese)

The principal features of the proposed theory are as follows. (1) Any three-dimensional elastic body in Euclidean space is considered to be bent at the yield-point into a curved state forming a Riemannian manifold. (2) The resistance to the sudden appearance of the Riemannian state can be related to a certain property due to the fine structure of the material. (3) The field equation of the equilibrium of the forces acting in the direction normal to the original flat manifold in the enveloping space is established by an application of the results of the author's paper reviewed above. (4) The equation obtained is of the same type as that in the theory of buckling of a flat plate. (5) Sudden yielding of a material such as mild steel will be obtained for a characteristic value of this equation under suitable boundary conditions. *A. Kawaguchi* (Sapporo).

Mann, Elizabeth H. An elastic theory of dislocations. Proc. Roy. Soc. London. Ser. A. **199**, 376–394 (1949).

An extension of the ideas of dislocation theory (following the work of Volterra) is presented to describe the internal stresses set up by slip processes in crystalline materials. A somewhat more general type of dislocation has been obtained by demanding less restrictive conditions on the stress components acting across a surface in the material. The discontinuities of displacement across a slip band are more general than in Volterra's work because some of the stress components are discontinuous. The theory presented includes Volterra's results. Four examples are worked out with plane and screw dislocations in cylinders. The dislocation theory is connected with simple types of slip in crystals. Replacing slip bands in a deformed crystal by dislocations allows calculation of stresses in the elastic region. *R. Truell* (Providence, R. I.).

Postacıoğlu, Bekir. Remarques sur les conditions d'existence des courbes intrinsèques. Bull. Tech. Univ. Istanbul **1**, 36–47 (1948). (French. Turkish summary)

According to Mohr's theory the normal stress τ and the shearing stress σ which will cause failure in an elastic material are connected by a continuous set of definite relations given by an intrinsic curve of the form $\tau=f(\sigma)$. Failure occurs when this curve touches the outermost of the Mohr circles. This curve can be obtained as an envelope of the family of circles $(x-p)^2+y^2=r^2$, where p and r are half the sum and difference of the principal stresses. Assuming that r is a function of p the author uses elementary methods to obtain the intrinsic curve. A linear relation between r and p gives good agreement with an empirical curve obtained in the case of concrete. This linear relation implies that, in the criterion of failure, besides the shearing stress, which is a primary factor in yield, volumetric stress of a particular type should also be taken into account. Such a law was formulated by J. J. Guest using various experimental re-

sults [Philos. Mag. (7) **29**, 289–297 (1940)]. B. R. Seth [Proc. Indian Acad. Sci., Sect. A. **14**, 37–40 (1941)] deduced it as a direct consequence of the theory of finite deformation.
B. R. Seth (Ames, Iowa).

Skudrzyk, E. Die innere Reibung und die Materialverluste fester Körper. I. Allgemeine Theorie. Österreich. Ing.-Arch. **3**, 356–373 (1949).

This paper attempts to treat elastic after-effects by the common expedient of superposing ordinary elastic and viscous effects [cf., e.g., Weissenberg, Abh. Preuss. Akad. Wiss. Phys.-Math. Kl. **1931**, no. 2]. The author's theory is tacitly limited to infinitesimal displacement gradients, since he does not distinguish between Eulerian and Lagrangian coordinates and since he employs local rather than material time derivatives. He proposes the familiar expression $t^i_j = f(\partial/\partial t)\xi^k{}_{,k}\delta^i_j + g(\partial/\partial t)(\xi^i{}_{,j} + \delta_{j,}{}^i)$, where t^i_j is the stress tensor, ξ^i is the displacement vector, and f and g are power series. He notes that (questions of convergence being neglected) in a harmonic oscillation f and g reduce to complex power series in the frequency, so that the response of the material may then be characterized by four real functions of frequency. He discusses the definition of elastic moduli and damping coefficients appropriate to various special cases in terms of these functions. A second part is to follow.
C. Truesdell (Washington, D. C.).

Weber, C. Berichtigung zur kleinen Mitteilung: C. Weber, Zur nichtlinearen Elastizitätstheorie. Z. angew. Math. Mech. 28 (1948), S. 189–199. Z. Angew. Math. Mech. **29**, 256 (1949).

Cf. these Rev. **10**, 169. The correct pagination of the article in question is 189–190.

Leibfried, Günther, und Lücke, Kurt. Über das Spannungsfeld einer Versetzung. Z. Physik **126**, 450–464 (1949).

The paper is well summarized by the following author's abstract. The stress field of a dislocation is described by various methods and the results are compared. In the first section a solution based on the theory of elasticity is given; in the second section a simplified version and generalization of Peierls' discussion is given which takes approximate account of the atomic structure. It turns out that these solutions practically agree. Finally these solutions are compared to that given by Taylor. *R. Truell.*

Levi, Franco. Generalizzazione del legame sforzi-deformazioni nei solidi elastico-viscosi. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 304–310 (1949).

The paper discusses a solid composed of fibres, some elastico-viscous and the others elastic, the fibres adhering to one another. The flow of such a solid under stress is described by an integro-differential equation, immediately reducible to a linear differential equation of the second order.
J. L. Synge (Dublin).

Castiglia, Cesare. Effetto dell'elasticità ritardata sugli stati di coazione di origine viscosa. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 311–314 (1949).

Using two different simple special laws for the behavior of an elastico-viscous fibre, the author solves the equation given by F. Levi in the paper reviewed above.
J. L. Synge (Dublin).

Bijlaard, P. P. Theory and tests on the plastic stability of plates and shells. J. Aeronaut. Sci. **16**, 529–541 (1949).

The author summarizes a very extensive series of his papers on plasticity dating back to 1931. In the present paper a comparison is given between tests and calculations of buckling load on the basis of deformation theory and fully plastic behavior. Bars, plates, and shells with various boundary and loading conditions are analyzed in a straightforward manner following the basic assumptions. Good agreement is reported for all cases. The presence of initial eccentricity or the Shanley concept of loading during buckling well justifies fully plastic behavior as contrasted with partial elastic unloading due to bending alone. However, the reviewer cannot agree that the success in matching these experimental results in itself permits the indiscriminate use of deformation theory in new applications.
D. C. Drucker (Providence, R. I.).

Drucker, D. C. A discussion of the theories of plasticity. J. Aeronaut. Sci. **16**, 567–568 (1949).

The author refers particularly to papers by Batdorf [same vol., 405–408 (1949)] and Bijlaard [see the preceding review].

Markov, A. A. On variational principles in the theory of plasticity. Grad. Div. Appl. Math. Brown Univ. Translation **A11-T-3**, 22 pp. (1948).

Translated from Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 339–350 (1947); these Rev. **9**, 120.

Phillips, Aris. Variational principles in the theory of finite plastic deformations. Quart. Appl. Math. **7**, 110–114 (1949).

Two useful minimum principles are established for large deformations. In the linear or nonlinear theory of small strain elasticity they are the familiar minimum potential and minimum complementary energy theorems. The author uses the more descriptive terminology of virtual displacement and virtual stress variation principles. However, the word plastic in the title is somewhat misleading as a deformation or total theory is postulated. This means that although the result applies to a material with a nonlinear elastic stress-strain relation, it is not valid when unloading occurs at any point in a plastic body. An extended von Mises criterion of loading is employed. *D. C. Drucker.*

Pastori, M. Plasticità. Rend. Sem. Mat. Fis. Milano **18** (1947), 93–113 (1948).

The paper is of expository character; it contains a brief survey of the mathematical theory of plasticity with special reference to Italian contributions to this theory.
W. Prager (Providence, R. I.).

Gol'denblat, I. I. Some general laws of a process of elastic-plastic deformation. Doklady Akad. Nauk SSSR (N.S.) **68**, 1005–1008 (1949). (Russian)

The author discusses the general mathematical structure of certain nonlinear stress-strain laws for isotropic plastic materials. These laws are all of the "finite" type, and the author does not seem to be aware of the fact that stress-strain laws of this type cannot claim general validity in the plastic range, but apply only under special circumstances when the more adequate "incremental" laws can be integrated to yield finite laws. *W. Prager.*

Hill, Rodney. General features of plastic-elastic problems as exemplified by some particular solutions. J. Appl. Mech. **16**, 295–300 (1949).

Basing his discussion on the stress-strain relations of Prandtl-Reuss, the author presents new complete solutions of the following elastic-plastic problems: uniform extension under condition of plane strain, expansion of a spherical shell by internal pressure, and expansion of a cylindrical hole by internal pressure (plane strain). These solutions are used to illustrate certain features common to all elastic-plastic problems. *W. Prager* (Providence, R. I.).

Hill, R. Plastic distortion of non-uniform sheets. Philos. Mag. (7) **40**, 971–983 (1949).

The author investigates the plastic straining of a flat sheet in its plane, assuming that, for practical purposes, the state of stress can be treated as plane. The thickness of the sheet may be nonuniform from the outset, or become so as a result of the plastic deformation, but the derivative of the sheet thickness in any direction of the middle plane of the sheet is supposed to be small compared with unity. The basic equations for the stress and velocity components are established and discussed by the method of characteristics. The expansion of a circular hole in an infinite plate by internal pressure is treated as an example; the results are compared with Taylor's theoretical and experimental treatment of the same problem. *W. Prager.*

Hill, R. The theory of plane plastic strain for anisotropic metals. Proc. Roy. Soc. London. Ser. A. **198**, 428–437 (1949).

The anisotropic quadratic yield criterion considered previously [same Proc. Ser. A. **193**, 281–297 (1949); these Rev. **10**, 83] is adopted, the strain-rate components being obtained from it by differentiation according to the concept of plastic potential. Combination of these relations for plane strain with the equations of equilibrium determines pairs of partial differential equations for stress and velocity components which are hyperbolic, the characteristics in both cases being the lines of zero rate of extension. Relations along the characteristics are developed, providing an extension of Hencky's geometrical laws for the isotropic case. The solution for indentation into a plane surface by a flat rigid die is given. *E. H. Lee* (Providence, R. I.).

Kačanov, L. M. On the theory of transient creep. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 381–390 (1949). (Russian)

Creep is treated as a process of nonlinear viscous flow. The viscous strain rate ϵ_{ij} is supposed to be related to the stress σ_{ij} and the stress rate τ_{ij} by means of

$$\epsilon_{ij}=B(t)T^{m-1}(\sigma_{ij}-\tfrac{1}{3}\sigma_{kk}\delta_{ij})+\tfrac{1}{2}G^{-1}(\tau_{ij}-\nu(1+\nu)^{-1}\tau_{kk}\delta_{ij}),$$

where δ_{ij} is the Kronecker delta, T a quadratic invariant of the stress deviation $\sigma_{ij}-\sigma_{kk}\delta_{ij}/3$, ν the Poisson ratio, G the shear modulus, m a constant, and $B(t)$ a monotonically decreasing function of time. A variational principle for materials of this type is established, and various examples are discussed including flexure and torsion of bars and the general relaxation problem where the displacements are prescribed on part of the surface of a body while the tractions are required to vanish on the remainder of the surface. *W. Prager* (Providence, R. I.).

Oldroyd, J. G. Rectilinear flow of non-Bingham plastic solids and non-Newtonian viscous liquids. I. Proc. Cambridge Philos. Soc. **45**, 595–611 (1949).

The paper is concerned with the flow of a viscous liquid or viscoplastic solid through a straight pipe of arbitrary cross section. If the z-axis is taken in the direction of flow, the shear stress vector in the cross sectional plane is assumed to have the direction of the gradient of the velocity $w(x, y)$; the intensity τ of the shear stress vector and the shear rate $\gamma=|\text{grad } w|$ are supposed to be related to each other by the parametric relations $\tau=\tau(\xi)$, $\gamma=\gamma(\xi)$. It is shown that when the velocity contours $w=$constant and their orthogonal trajectories $W=$constant are taken as the coordinate lines of a system of curvilinear coordinates in the cross sectional plane, the line element can be written in the form $ds^2=[dw/\gamma(\xi)]^2+[dW/\tau(\xi)]^2$, where ξ as function of w and W satisfies the partial differential equation

$$\frac{\partial}{\partial w}\left\{\frac{\gamma\tau'}{\tau^2}\frac{\partial\xi}{\partial w}\right\}+\frac{\partial}{\partial W}\left\{\frac{\tau\gamma'}{\gamma^2}\frac{\partial\xi}{\partial W}\right\}=0$$

which is discussed at some length. When a particular real solution $\xi(w, W)$ of this equation is known, the curvilinear coordinate system can be constructed. Since this differential equation is more readily solved for some materials than for others, the author proposes that those materials associated with a more tractable differential equation be used as prototype materials for further theoretical discussion. *W. Prager* (Providence, R. I.).

Sauer, Robert. Über die Gleitkurvennetze der ebenen plastischen Spannungsverteilungen bei beliebigem Fliessgesetz. Z. Angew. Math. Mech. **29**, 274–279 (1949). (German. Russian summary)

The paper contains a discussion of two-dimensional problems of plasticity which is patterned on the Prandtl-Busemann method of treating plane supersonic flows. The author does not seem to be aware of similar work by V. V. Sokolovskiĭ [Theory of Plasticity, Izdatel'stvo Akad. Nauk SSSR, Moscow-Leningrad, 1946; these Rev. **8**, 545]. *W. Prager* (Providence, R. I.).

Sokolovskiĭ, V. V. On a plane problem of the theory of plasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 391–400 (1949). (Russian)

The paper is concerned with states of plane strain in a plastic medium which obeys a yield condition of the type proposed by O. Mohr. If σ_1 and $\sigma_2\leqq\sigma_1$ denote the principal stresses in the plane of strain, the yield condition is assumed to have the form $\frac{1}{2}(\sigma_1-\sigma_2)/k=\sin\{\frac{1}{2}(\sigma_1+\sigma_2)/k+H/k\}$, where H and k are constants and $\sigma_1+\sigma_2$ is assumed to satisfy $-2H\leqq\sigma_1+\sigma_2\leqq-2H+k\pi$. *W. Prager.*

Swida, W. Über die Formänderungen der Balken im elastisch-plastischen Zustand. Ing.-Arch. **17**, 71–87 (1949).

It is well known that the deflection of an elastic beam can be obtained as the bending moment of the conjugate beam under a distributed load which is given by the reduced bending moment of the original beam (i.e., bending moment divided by bending stiffness). The author points out that, for statically determinate beams at least, this method of determining the deflection is readily extended to elastic-plastic beams by defining the reduced bending moment in a suitable manner. The relation between the actual bending moment and the reduced moment is discussed for various

structural sections, and the elastic-plastic deflections of simply supported beams and cantilever beams under various types of loading are determined. *W. Prager.*

Stowell, Elbridge Z. A unified theory of plastic buckling of columns and plates. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 898, 11 pp. (1948).

Formerly issued as Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1556 (1948); these Rev. 10, 82.

Stowell, Elbridge Z., and Pride, Richard A. Plastic buckling of extruded composite sections in compression. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1971, 15 pp. (1949).

Negoro, Shosaburo, and Sekiya, Tsuyoshi. On a method of solving the so-called elastic plane stress problems. J. Osaka Inst. Sci. Tech. Part I. 1, 37–60 (1949).

The authors discuss the general plane stress problem of a plate with holes. They show that Airy's stress function is a many-valued plane biharmonic function when the domain considered is multiply-connected and when the forces and twisting moments on the respective boundaries are not in equilibrium. The functions are shown to be separable into two parts, (a) a known biharmonic function in the form of a series of arctangents and (b) an unknown single-valued biharmonic function. It is shown that the unknown function can always be found by numerical methods. It is further shown that the plane biharmonic function can be approximately rewritten by a two-dimensional interpolation formula. The errors involved by using a difference equation for the biharmonic equation are discussed as is also the convergence of the iteration method. The relaxation method is shown to be applicable.

Finally, the authors compare their method with one due to F. Wolf [reference not given] by considering the case of a square plate with a square hole, the outer boundary being subjected to normal loads.

H. D. Conway (Ithaca, N. Y.).

Olszak, W. Plane strain and plane stress systems. Arch. Méc. Appl., Gdańsk 1, 255–263 (1949).

It is known that the solution of generalised plane stress problems can be deduced from the corresponding solution for plane strain by changing the Poisson's ratio σ into the plane stress ratio σ_1 such that $(1+\sigma)(1-\sigma_1)=1$. By making σ_1 a function of z, the axial coordinate, stress-strain relations are generalised and plane strain and generalised plane stress problems become limiting cases. The corresponding components of displacements are obtained and the case of a single force acting within an infinite plate is discussed in detail. *B. R. Seth* (Ames, Iowa).

Položiĭ, G. N. Solution of the third basic problem of the plane theory of elasticity for an infinite plane with a square opening. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 297–306 (1949). (Russian)

The third fundamental boundary value problem of plane theory of elasticity is concerned with the determination of stresses in the interior of a two-dimensional region R when the tangential stress is specified on the boundary C of R and when displacements in the direction of the normal line to C are known. This problem was solved by N. I. Muschelishvili [C. R. (Doklady) Acad. Sci. URSS (N.S.) 4 (1934 III), 141–144] for regions R that can be mapped conformally on a unit circle by means of rational functions. In this paper the third boundary value problem is solved for an infinite plane region bounded by a square. The solution is obtained by reduction to the solution of known boundary value problems in the theory of functions of a complex variable. To ensure the existence of a solution it is necessary to impose certain hypotheses concerning the rate of growth of stresses in the neighborhood of the angular points of the region. *I. S. Sokolnikoff.*

Suhara, Toyotarō. On the stresses in a circular disc of variable thickness, the coefficients of elasticity and of the thermal expansion of the material being taken as functions of temperature. Proc. Fac. Engrg. Keiogijuku Univ. 1, 43–46 (1948). (Japanese. English summary)

The stresses due to the rotation and the thermal expansion of a thin circular disc of variable thickness z, which is expressed by $z=z_0(1-ar^r)^k$, are analysed for given temperature distributions, $T=T_0(1+br^r)$. The coefficient of thermal expansion α and that of elasticity are taken as functions of temperature, or as functions of the distance r from the axis of rotation, such as $\alpha=\alpha_0(1+b'r^r)^k$ and $E=E_0(1-cr^r)^j$.

From the author's summary.

Sen, Bibhutibhusan. Note on the deformation produced by some symmetrical distribution of variable loads on the plane boundary of a semi-infinite elastic solid. Bull. Calcutta Math. Soc. 41, 77–82 (1949).

A simple general method is given for solving the problem of a semi-infinite isotropic elastic solid under the action of a load distributed uniformly over any area on its plane boundary. The general method may be applied to a non-symmetrical load distribution, but the author only gives the details for a symmetrical distribution over a circular area, which is such that the load varies as a function $f(r)$ of the distance r from the centre of the circle. The particular paraboloidal distribution for which $f(r)=1-r^2/a^2$ over a circular area of radius a is given in full detail, while results for two load distributions varying exponentially with r have been stated briefly. *R. M. Morris* (Cardiff).

Hopkins, H. G. Elastic deformations of infinite strips. Proc. Cambridge Philos. Soc. 46, 164–181 (1950).

The Fourier integral solution of the biharmonic equation is used to solve some problems of generalized plane stress and bending of an infinitely long rectangular strip with mixed boundary conditions. Numerical results have been obtained for the case of a concentrated force. The analysis may be simplified by using complex function theory methods and the conformal transformation $\zeta=\exp(\pi z/a)$ which maps the strip into a semi-infinite plane with a straight boundary. *B. R. Seth* (Ames, Iowa).

Nowacki, W. Flexion des plaques continues infiniment longues. Arch. Méc. Appl., Gdańsk 1, 173–189 (1949). (Polish. French summary)

Dans le présent mémoire l'auteur a donné la solution exacte du fléchissement d'une plaque continue infiniment longue au moyen de la formule intégrale de Fourier. La résolution du problème a été réduite aux équations de trois moments dans la méthode des forces et aux équations de trois angles dans la méthode de déformation.

Author's summary.

Nowacki, W. The bending of compressed continuous plate. Arch. Méc. Appl., Gdańsk **1**, 67–79 (1949). (Polish. English summary)

The author presents the solution of the partial differential equation governing the deflection w of a rectangular isotropic plate loaded uniformly with the load p and compressed by a load q distributed uniformly along the edges in the plane of the plate. He discusses three cases: (1) a plate of infinite length compressed longitudinally; (2) a plate of infinite length compressed transversally; (3) a rectangular plate simply supported, compressed by a concentrated force p. The solutions, which are in the form of trigonometric series, lead to the conclusion that the plate will be destroyed, i.e., the stresses will be in the plastic region, before p will reach its critical value for buckling and only for very small compressive loads q will the load p approach the critical limit. *T. Leser* (Lexington, Ky.).

***Nowacki, Witold. The bending of a compressed continuous plate.** Proc. Seventh Internat. Congress Appl. Mech., 1948, v. 1, pp. 160–173.

This appears to be a translation of the paper reviewed above. *H. D. Conway* (Ithaca, N. Y.).

Green, A. E. The elastic equilibrium of isotropic plates and cylinders. Proc. Roy. Soc. London. Ser. A. **195**, 533–552 (1949).

The paper deals with three-dimensional effects in the static bending or stretching of isotropic elastic plates. The components of displacement and stress are expressed in terms of Boussinesq's potentials and very general solutions are obtained in the form of trigonometric and power series in the coordinate normal to the plane faces of the plate. The coefficients of the series are functions of the other two coordinates and are governed by ∞^3 second order partial differential equations which the author indicates are sufficient to satisfy three boundary conditions at each point of two cylindrical surfaces whose generators are normal to the plane faces. For the case of bending, a first approximation in terms of stress resultants and stress couples is shown to be almost identical with E. Reissner's theory [Quart. Appl. Math. **5**, 55–68 (1947); these Rev. **8**, 547]. No boundary value problems are considered for the general theory, but the approximate bending equations are solved for pure flexure of an infinite plate containing a circular cylindrical hole and the results are compared with Reissner's [J. Appl. Mech. **12**, A-69–A-77 (1945); these Rev. **7**, 42].

R. D. Mindlin (New York, N. Y.).

Green, A. E. On Reissner's theory of bending of elastic plates. Quart. Appl. Math. **7**, 223–228 (1949).

Among the methods for deriving two-dimensional theories of bending of elastic plates are those which use the calculus of variations, following Lagrange, and those in which the three-dimensional equations are integrated over the thickness of the plate, a method used by Boussinesq, at the suggestion of de St. Venant [Saint-Venant and Flamant, Théorie de l'Élasticité des Corps Solides de Clebsch, Dunod, Paris, 1883, p. 689]. E. Reissner, using his own version of the former method, has recently arrived at an improvement over the classical theory by including a first approximation of the influence of transverse shear deformation [see the reference given in the preceding review]. The author has shown previously [see the preceding review] that a portion of his three-dimensional theory of plates is almost identical with Reissner's theory. He now shows how Reissner's theory may be obtained directly by Boussinesq's method. Following this, Reissner's "equivalent rotations" are expressed in terms of functions corresponding to the potentials which give rise to the dilatation and rotation in the three-dimensional theory. A general solution of the simultaneous equations governing the dilatational potential and Reissner's "weighted average displacement" is given in terms of functions of a complex variable and a function whose double Laplacian is the transverse load. *R. D. Mindlin.*

***Reissner, Eric. On finite deflections of circular plates.** Proc. Symposia Appl. Math., Vol. I, pp. 213–219. American Mathematical Society, New York, N. Y., 1949. $5.25.

The complete theory of rotationally symmetric deformations of circular thin plane elastic plates is worked out assuming all strains to be small. However, it is expressly not assumed that the deflections and slopes of the strained middle surface of the plate remain small after the deformation. Nevertheless the theory results in differential equations which are hardly less tractable than those obtained assuming small slopes and deflections, to which they of course reduce when the appropriate assumptions are made. It is pointed out that boundary layer effects occur at the edge of the plate and that they can be treated by methods similar to those used by Friedrichs and Stoker [Amer. J. Math. **63**, 839–888 (1941); these Rev. **3**, 223], who worked with the theory which assumes small slopes of the middle surface of the plate; and this in turn has the advantage of yielding more accurate results since the slope at the edge may become large for plates of small thickness even though the strains remain small. *J. J. Stoker.*

Conway, H. D. Bending of rectangular plates subjected to a uniformly distributed lateral load and to tensile or compressive forces in the plane of the plate. J. Appl. Mech. **16**, 301–309 (1949).

The problem of a simply supported rectangular plate bent by uniform lateral load and tensile or compressive forces acting perpendicular to a pair of opposite edges is solved by conventional methods, and graphs are drawn to calculate the maximum deflection and stress. It may be pointed out that by extending the method [cf. B. R. Seth, Bull. Calcutta Math. Soc. **40**, 36–40 (1948); these Rev. **10**, 172] which makes such problems depend upon the corresponding torsion solution, the problem for a tension or a compression acting uniformly in all directions can be readily solved and the results extended to boundaries other than those which are rectangular. *B. R. Seth.*

Grinberg, G. A., and Uflyand, Ya. S. On the bending of a rectangular plate with a clamped boundary under an arbitrary load. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 413–434 (1949). (Russian)

The existing solutions of the problem of small deflections of a clamped rectangular plate under an arbitrary normal load lead to laborious calculations for the bending moments and shearing forces. This paper is concerned with the development of computationally more tractable formulas. In section 1 of the paper the boundary value problem is reduced to the solution of an integral equation, which is solved approximately. In sections 2 and 3, the approximate solution deduced in section 1 is used to construct practically useful formulas for deflection, bending moments and shearing forces for several special types of loading.

I. S. Sokolnikoff (Los Angeles, Calif.).

Sapondžyan, O. M. Application of the method of additional reactions to the solution of the problem of bending of plates, the plane problem and the problem of torsion of prismatic bars. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 501–512 (1949). (Russian)

The paper contains several examples of solution of the boundary value problems in isotropic elasticity by a procedure termed by the author "the method of additional reactions." The method depends on ingenious devices suitable to a particular problem under consideration. The essence of the idea is to replace the consideration of a given boundary value problem by another one involving the region whose boundary, in part, coincides with the original one, and introduce in the solution of this new problem sufficient parameters in order to satisfy the boundary condition of the original problem. The idea can be illustrated by the problem of small deflections of a thin clamped semicircular plate under a uniform normal load p per unit area of the plate. The clamping along the diameter of the semicircle introduces unknown shearing forces. Now, consider a clamped circular plate of the same radius as the original one, which is subjected to a load p and to a line distribution of load of intensity q along the diameter. If the diameter coincides with the y-axis of Cartesian coordinates, and the distribution q is taken in the form $\sum_{n=0}^{\infty} q_{2n} y^{2n}$, the formal solution for the deflection w is obtained in the form $w=w(x, y, q_{2n})$, where the q_{2n} are parameters. These parameters can be selected so as to satisfy the condition $\partial w(0, y)/\partial x=0$. The function w so obtained satisfies the condition of clamping along the diameter of the semicircle in the original problem. This device is applied to several plate deflection problems, the torsion problem for a rectangle with a slit, and the plane problem of elasticity for a semicircle fixed rigidly along the diameter and subjected to a uniform radial pressure along the curved boundary.

I. S. Sokolnikoff (Los Angeles, Calif.).

Sengupta, H. M. On the bending of an elastic plate. I. Bull. Calcutta Math. Soc. 41, 163–172 (1949).

The author solves the problem of a thin elliptic plate with a clamped edge, bent by a concentrated load acting at any point on its surface. No numerical results are given.

H. D. Conway (Ithaca, N. Y.).

Mitra, D. N. On the flexure problem of a limacon and some other boundaries. Bull. Calcutta Math. Soc. 41, 153–158 (1949).

Asserting that Stevenson's result for the center of flexure of a limaçon [Proc. London Math. Soc. (2) 45, 126–143 (1939)] is incorrect, the author derives another result which does not contain the known result for a cardioid by a factor 2. However, the author asserts that Stevenson's result is in error by use of an unnecessary factor 2 arising early in Stevenson's paper. [The reviewer has not verified these errors.] As a second example the flexure problem for a cylinder whose cross section is the inverse of an ellipse is treated and is said to agree with known results.

D. L. Holl (Ames, Iowa).

Ōkubo, H. On the problem of a notched plate of an aeolotropic material. Philos. Mag. (7) 40, 913–917 (1949).

This paper is a continuation of a series of papers by the same author. In Cartesian coordinates a stress function is expressed in terms of two functions of two complex variables. The Cartesian coordinates are then transformed into two suitably chosen systems of curvilinear coordinates which coincide on the bounding curves. The way in which the Poisson's ratios and the modulus of rigidity in the plane of the plate enter in the analysis is not indicated in the present paper. This point may be clarified in earlier papers of the series. The same problem for a plate of orthotropic material has been solved by A. E. Green [Proc. Roy. Soc. London. Ser. A. 184, 289–300 (1945); these Rev. 7, 268] and by C. B. Smith [Quart. Appl. Math. 6, 452–456 (1949); these Rev. 10, 341].

H. W. March (Madison, Wis.).

Tarabasov, N. D. Strength of shrink-fitted plates. Doklady Akad. Nauk SSSR (N.S.) 67, 615–618 (1949). (Russian)

The method of solution of plane elastostatic problems developed by Muskhelišvili is used to solve the following problem. A thin circular plate is formed by shrink-fitting concentric circular rings onto a circular disk. The elastic properties of the disk and rings are identical and the material is isotropic. The external boundary of this composite circular plate is subjected to the action of distributed self-equilibrating forces lying in the plane of the plate and to a distribution of tangential forces giving rise to a torsional couple. The torsional couple is balanced by the moment applied to the central disk. The discontinuities along the boundaries of concentric rings are specified. These, in general, are unequal and are determined by the magnitudes of the differences in the radii of adjoining rings before they are shrink-fitted on the central disk.

I. S. Sokolnikoff.

Dokos, S. J. A force applied in the median plane at the center of a circular insert in a plate. J. Appl. Mech. 16, 411–413 (1949).

The problem described in the title is solved by using, for each region, a stress function consisting of two terms. Each term is a solution of the biharmonic equation. The first term gives the well-known solution to the problem of a concentrated force acting in the median plane of an infinite plate. The second terms for each region are chosen to meet the continuity conditions for the disk and plate without violating the other boundary conditions. [There is a typographical error in equation (4) of the paper, the second term on the right-hand side of which should be $\psi_i(r, \theta)$.]

H. D. Conway (Ithaca, N. Y.).

Woinowsky-Krieger, S. Berechnung einer auf elastischem Halbraum aufliegenden, unendlich erstreckten Platte. Ing.-Arch. 17, 142–148 (1949).

The author points out the advantages of treating some problems in soil mechanics and allied subjects by using the analogy of an infinite elastic plate resting on a semi-infinite elastic solid. Employing the usual assumptions of the thin plate theory and the results of his previous paper [Ing.-Arch. 3, 236–250 (1932)] he obtains numerical values for the deflection, pressure and stress when the plate is acted upon by (1) a point load, (2) a circular line load. A comparison is also made with the results given by the Hertz theory.

B. R. Seth (Ames, Iowa).

Laura, Ernesto. Una osservazione sopra l'equilibrio delle superficie rigate sviluppabili flessibili ed inestendibili. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 99, 339–356 (1940).

The author is concerned with the solution of Beltrami's equations [Opere Matematiche, v. 3, pp. 420–464 = Mem. Accad. Sci. Ist. Bologna (4) 3, 217–265 (1882)] for the equilibrium of a loaded flexible and inextensible sheet. He

considers only the case when the form of the sheet is a developable surface. By vectorial transformations he obtains finite expressions for the Lagrangian multipliers (which are closely related to the stress components, of which but two are nonvanishing and unequal to each other) in terms of two arbitrary functions. Elimination of these two functions yields the differential equations of the problem. This elimination is in general an elaborate process, and the author restricts his further analysis to two special types of support for a sheet subject to gravity: (a) two equal parallel line segments, the initially plane sheet being of a shape symmetrical with respect to the line joining their middle-points; (b) two intersecting line segments. In case (a) the author assumes that the sheet takes on a cylindrical form, and reduces the problem of finding this form to a quadrature. In case (b) he assumes the form to be conical, and reduces the problem of finding it to the integration of a complicated nonlinear ordinary differential equation. *C. Truesdell.*

Burmistrov, E. F. Symmetrical deformation of a nearly cylindrical shell. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 401–412 (1949). (Russian)

The method of small parameters and the approximate Kirchhoff-Love theory of shells are used to study symmetrical deformations of nearly cylindrical orthotropic shells of revolution with the radius of cross-section of the form $r=a+\lambda f(x)$, where λ is a small parameter. The main part of the paper is concerned with the deformation of shells with parabolic meridional section, given by $r=a[1+\lambda(x^2/l^2-1)]$, $2l$ being the length of the shell. It is assumed that the shell is deformed by (a) uniform axial forces applied along the edge of the shell, (b) normal forces uniformly distributed over its lateral surface. In the application of the method of perturbations it is assumed that the displacements are of the form $u=u_0+\lambda u_1+\lambda^2 u_2$, and the stresses and moments are likewise given with sufficient accuracy by similar expressions. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Vekua, I. N. On the theory of elastic shells. Doklady Akad. Nauk SSSR (N.S.) **68**, 453–455 (1949). (Russian)

The author gives a complex form of the basic equations of the shallow elastic shell theory when the coordinate net is orthogonal isometric. *I. S. Sokolnikoff.*

Vlasov, V. Z. Contact problems in the theory of shells and thin rods. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk **1949**, 819–836 (1949). (Russian)

The article contains a condensed account [based on papers and books by the author] of the application of the engineering theory of shells to the stress analysis of thin circular cylinders so reinforced by longitudinal and transverse ribs that the elastic system can be considered orthotropic.

I. S. Sokolnikoff (Los Angeles, Calif.).

Huber, M. T. Bending of the curved tube of elliptic section. Arch. Méc. Appl., Gdańsk **1**, 1–22 (1949). (Polish. English summary)

The theory is based on the following assumptions. (1) The thickness of the tube's partition is small in comparison with the axes $2a$ and $2b$ of the elliptic section; (2) the radius of curvature of the tube's axis R is constant and large in comparison with a and b; (3) a is perpendicular to R; (4) by the bending the elliptic section is transformed into another elliptic section without a significant change in the length of the circumference of the middle line. By using the principle of minimum elastic energy there is obtained a formula giving the dependence of the relative variation of the initial angle between two cross sections on the bending moment.

From the author's summary.

Batdorf, S. B. A simplified method of elastic-stability analysis for thin cylindrical shells. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. **874**, 25 pp. (1947).

Formerly issued as Tech. Notes Nat. Adv. Comm. Aeronaut., nos. **1341**, **1342** (1947); these Rev. **8**, 548.

Langhaar, H. L. A strain-energy expression for thin elastic shells. J. Appl. Mech. **16**, 183–189 (1949).

Reissner, Eric. Note on the problem of twisting of a circular ring sector. Quart. Appl. Math. **7**, 342–347 (1949).

The author develops formulas for the shear stress resultant and the twist-torque ratio in close coiled helical springs with hollow cross sections. These are analogous to the formulas of Bredt for the St. Venant torsion of cylindrical rods with thin-walled hollow sections. Examples of circular and rectangular cross sections with uniform wall thickness are treated. *D. L. Holl* (Ames, Iowa).

Ufiyand, Ya. S. The torsion of a prismatic bar with a profile bounded by arcs of two intersecting circles. Doklady Akad. Nauk SSSR (N.S.) **68**, 17–20 (1949). (Russian)

Bipolar coordinates are used to obtain a solution of Saint Venant's torsion problem for a region bounded by two intersecting circular arcs. *I. S. Sokolnikoff.*

Abramyan, B. L. The torsion of prismatic bars with cruciform cross section. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 551–556 (1949). (Russian)

The paper contains a solution of Saint Venant's torsion problem for a bar of cruciform cross-section. Prandtl's torsion function is obtained in the form of an infinite series involving hyperbolic and trigonometric functions. The computation of torsional rigidity, based on the Prandtl function, is compared with results given by C. Weber's empirical formulas [Die Lehre der Drehungsfestigkeit, Berlin, 1921] and found to be in good agreement for thin-flanged sections only. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Parkus, H. Die Torsion der Kreiswelle mit rechteckiger Längsnut. Österreich. Ing.-Arch. **3**, 336–344 (1949).

The torsional problem for a circular shaft with a rectangular keyway is treated by means of conformal representation. Instead of the more usual function ψ conjugate to the torsional function φ, which simply reduces to $-(x^2+y^2)+$constant on any surface free from applied traction, the author uses a different function, the boundary condition for which is expressed in terms of its gradient along the curve, and takes different forms on the two portions of the shaft corresponding to the keyway and the remaining circular section. The reviewer thinks that as a result the solution of the problem is made more difficult. The flexure and torsion of a similar type of cylinder, namely one whose cross section is bounded by arcs of two orthogonal circles, has been examined by Wigglesworth and Stevenson [Proc. Roy. Soc. London. Ser. A. **170**, 391–414 (1939)], using the more usual function ψ for the torsion, and the Poisson integral method, but the author is apparently unacquainted with this work.

Having found the required function, the stress along the keyway is determined and in the numerical calculations it is found to attain its greatest value in the neighbourhood of

the corner. If the corner is sharp the maximum stress is of course infinite, but decreases rapidly with increasing radius of curvature at the corner; it always remains, however, more than twice the stress in a shaft without a keyway.

R. M. Morris (Cardiff).

Bogunovič, Vladimir. Sur la torsion d'un tube rectangulaire. Godišnjak Tehn. Fak. Univ. Beograd. **1946–47**, 23–25 (1949). (Serbian. French summary)

Hajdin, Nikola. Torsion of a triangular tube. Godišnjak Tehn. Fak. Univ. Beograd. **1946–47**, 27–29 (1949). (Serbian. Russian summary)

Lehnickiĭ, S. G. The stress distribution in an elastic bar with curvilinear anisotropy under the action of a stretching force and bending moments. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 307–316 (1949). (Russian)

A cylindrical rod of arbitrary cross section is made of elastic material possessing cylindrical anisotropy. The axis of anisotropy, in general, does not coincide with the axis of the cylinder, but the planes of elastic symmetry are normal to the generators of the cylinder. The lateral surface of the rod is free of external forces and the distribution of forces on the ends of the rod is statically equivalent to a single force directed along the axis of the rod, and a bending couple. A long isotropic rod subjected to such a distribution of forces experiences compression and pure bending, but in a long anisotropic rod in addition to bending and compressive stresses new stresses arise. These are typical of those that appear in the study of a plane deformation. The author's formulation of the problem of compression and bending of rods with cylindrical anisotropy leads him to the determination of the stress function satisfying a fourth order partial differential equation similar to that appearing in the analysis of plane deformation in anisotropic elasticity. The problem of compression of thick circular pipes by longitudinal forces is solved explicitly. One peculiarity of the solution is that for certain values of the elastic coefficients a concentration of stress appears in the neighborhood of the axis of anisotropy. The problem of bending of thick circular pipes by terminal couples is also completely solved for one case of a cylindrically orthotropic medium.

I. S. Sokolnikoff (Los Angeles, Calif.).

Conrad, Karl Leroy. Stress distribution due to hydrostatic pressure on a parabolic boundary. Iowa State Coll. J. Sci. **23**, 397–404 (1949).

The author obtains a solution in closed form to the problem of a semi-infinite isotropic cylinder with a parabolic boundary subjected to hydrostatic pressure, by using the complex function theory method. Stress distributions are plotted for a specific example. *H. D. Conway.*

Edelman, F. On the compression of a short cylinder between rough end-blocks. Quart. Appl. Math. **7**, 334–337 (1949).

A test specimen in the form of a right circular cylinder of radius b and altitude $2h$ is assumed to be compressed an amount $2a$ between parallel end-blocks, by a compressive force F. It is assumed that the curved surface of the cylinder is free of stress and that the plane ends undergo only normal displacements which in turn are uniform. The methods of Prager and Synge [same Quart. **5**, 241–269 (1947); these Rev. **10**, 81] are employed [cf. Greenberg and Truell, same Quart. **6**, 53–62 (1948); these Rev. **10**, 81, for the corresponding problem with a rectangular bar as test specimen]. It is found that, for the case $b=h=1$ and $\nu=\frac{1}{3}$, the value of Young's modulus is given by $E=0.9477\sigma/\epsilon$ in terms of the mean stress $\sigma=F/(\pi b^2)$ and mean strain $\epsilon=a/h$, the numerical coefficient being in error by less than 0.7 percent. *G. B. Thomas, Jr.* (Cambridge, Mass.).

Elliott, H. A. Axial symmetric stress distributions in aeolotropic hexagonal crystals. The problem of the plane and related problems. Proc. Cambridge Philos. Soc. **45**, 621–630 (1949).

The author applies earlier results [same Proc. **44**, 522–533 (1948); these Rev. **10**, 167] on the general solution of problems of stress systems in aeolotropic hexagonal crystals to "the problem of the plane" and related problems in the case of axial symmetry. The method of Hankel transforms developed by Harding and Sneddon [same Proc. **41**, 16–26 (1945); these Rev. **6**, 251] for isotropic materials is extended to the more general case. The solution to the problem can be found by determining functions $\phi_1(r, z)$, $\phi_2(r, z)$ satisfying given differential equations. If these functions are written as

$$\phi_i=\int_0^\infty \xi G_i(\xi z_i)J_0(\xi r)d\xi, \qquad i=1, 2,$$

the G_i are determined to within functions of ξ. The latter are found from the boundary conditions which lead to integral equations solvable by the methods of Titchmarsh and Busbridge.

The author gives the general solution for the indentation problem and applies this to the case of the rigid conical punch, the rigid spherical indentation, and the cylindrical punch. Similar results are obtained for the three-dimensional solid in which there is initially a thin disk-shaped crack in the plane $z=0$. The solution is carried out in detail for a disk-shaped "Griffith" crack in a solid under tension. The results for the isotropic solid can be deduced from this solution. *G. H. Handelman* (Pittsburgh, Pa.).

Walczak, J. La tension tangentielle sur une section inclinée. Arch. Méc. Appl., Gdańsk **1**, 191–202 (1949). (Polish. French summary)

La tension tangentielle (sur une plane de section inclinée par rapport aux trois sections perpendiculaires entre eux, passant par un point examiné et prises comme les planes de coordonnées) est présentée d'habitude dans la littérature comme une différence géometrique de la tension résultante et la tension normale. Dans le travail présent, on a essayé de trouver une formule générale pour la tension tangentielle sur une section inclinée, comme une fonction de six composantes, qui détermine la tension.

From the author's summary.

Naleszkiewicz, J. On the cooperation of two cantilever spars with a shear-resisting skin. Arch. Méc. Appl., Gdańsk **1**, 143–172 (1949).

Mitra, D. N. On flexure functions of a semi-circular cylinder. Bull. Calcutta Math. Soc. **41**, 125–128 (1949).

The author employs function-theoretic methods introduced by S. Ghosh [same Bull. **39**, 1–14 (1947); these Rev. **9**, 256]. The canonical flexure functions are determined for a semi-circular cylinder according to the method employed in previous papers [same Bull. **40**, 173–182 (1948); these Rev. **10**, 650] and are said to agree with those of Ghosh in an earlier paper [same Bull. **40**, 77–82 (1948); these Rev. **10**, 340]. *D. L. Holl* (Ames, Iowa).

Klitchieff, J. M. Buckling of continuous beams on elastic supports. Quart. J. Mech. Appl. Math. **2**, 257–262 (1949).

The author considers a column with simply-supported ends, and with a series of lateral elastic supports spaced equally along the column length. By expressing the buckling deflection of the column in the form of a trigonometric series, the author is able to derive a particularly simple expression for the stiffness required of the lateral supports to cause these support points to appear as nodes in the lowest buckling mode (i.e., the mode with the smallest critical load). *M. Goland* (Kansas City, Mo.).

Hlitčijev, J. Buckling of continuous beams on elastic supports. Glas Srpske Akad. Nauka **195**, 37–44 (1949). (Serbian)

Cf. the preceding review.

Hopkins, H. G. Elastic stability of infinite strips. Proc. Cambridge Philos. Soc. **45**, 587–594 (1949).

The author discusses the instability of an infinitely long rectangular plate (the long edges of which are simple supported) under the action of loads applied normally to the long edges. Neglecting the extensional deformations of the middle surface of the plate and using a strain-energy method, he shows that the critical loadings result from the eigenvalues of an ordinary differential equation or, alternatively, of an integral equation. The following special cases of loading are then treated: single concentrated compressive forces, two equal concentrated compressive forces, three equally spaced compressive forces, two equal and opposite forces, uniformly distributed forces. Numerical results are given for these cases. *H. D. Conway* (Ithaca, N. Y.).

Milosavljević, Miodrag. Sur la stabilité des plaques rectangulaires renforcées par des raidisseurs. Godišnjak Tehn. Fak. Univ. Beograd. **1946–47**, 127–142 (1949). (Serbian. French summary)

L'auteur à donné une solution de ce problème dans le cas d'une plaque rectangulaire renforcée par des raidisseurs horizontaux et verticaux, sollicitée simultanément à la flexion, au cisaillement et à un effort axial. Il a basé ses études sur l'analyse de l'équation différentielle de surface flambée de la plaque. *From the author's summary.*

Hemp, W. S., and Griffin, K. H. The buckling in compression of panels with square top-hat section stringers. Coll. Aeronaut. Cranfield. Rep. no. **29**, i+13 pp. (5 plates) (1949).

In order to obtain an approximate solution to the problem described in the title, the authors regard a stiffened panel as consisting of a number of flat plates joined together at their edges. The usual theory is then used to relate edge moments and slopes, and the resulting equations yield a relationship between the compressive stress and the half wave length λ in the direction of compression. The value of λ to give the least compressive stress is then found, and this stress is the buckling stress. In order to investigate the effect on the buckling stress of rounding the corners of the stringers, use is made of the theory of cylindrical shells. It is decided that this effect can be neglected. However, no account is taken of the stiffening effect of the stringer flanges. The results are compared with experiments. Reasonable agreement is obtained. *H. D. Conway* (Ithaca, N. Y.).

Cox, H. L., and Riddell, J. R. Buckling of a longitudinally stiffened flat panel. Aeronaut. Quart. **1**, 225–244 (1949).

By conventional energy methods, the authors investigate the conditions for buckling of a longitudinally stiffened flat panel, on the assumption that the rotational stiffness of the stringers is negligible. The analysis is applied to the cases of panels with one, two, or three stringers, with the objective of determining the limiting characteristics of the stringers to insure that these members remain straight up to the buckling stress of the sheet between stringers. The principal interest of the paper derives from the careful interpretation of the results, which affords convenient criteria for use by the designer. *M. Goland* (Kansas City, Mo.).

Hunter-Tod, J. H. The elastic stability of sandwich plates. Coll. Aeronaut. Cranfield. Rep. no. **25**, 42 pp. (13 plates) (1949).

This paper treats the elastic stability of supported rectangular plates of sandwich construction under compression and shear loading. Differential equations for the buckling process and formulae for critical stresses are developed for flat and curved panels in compression and the flat panels in shear. Buckling of the whole panel as well as wrinkling of the faces is considered. A series of diagrams gives the results of numerical evaluation of the author's formulae.

E. Reissner (Cambridge, Mass.).

Müller-Magyari, F. Kritische Spannungen dünnwandiger Plattenwerke unter zentrischem Druck. Österreich. Ing.-Arch. **3**, 180–196 (1949).

Previous work by the author [same Arch. **2**, 331–346 (1948); these Rev. **10**, 218] on local buckling of thin-walled axially loaded columns is extended. Now the free longitudinal edges of open sections are permitted to deflect but all other longitudinal edges, as before, are assumed to remain parallel to the axis. The standard form of solution for lateral displacement of thin flat rectangular plates is expanded in powers of the inverse of the square of the longitudinal wave length. Curves are drawn for the critical stresses in a plate of which one longitudinal edge is free or flanged and the other is elastically restrained. Useful curves are also given for several open sections.

D. C. Drucker (Providence, R. I.).

Wierzbicki, W. Arithmétisation des problèmes de flambage. Arch. Méc. Appl., Gdańsk **1**, 23–66 (1949).

This is an expository article. It illustrates applications of the iterative method of Vianello and Stodola and of the method of finite differences to problems of buckling of beams, plates and arches. *G. B. Thomas, Jr.*

Wierzbicki, W. The quasi-unstable equilibrium in structural mechanics. Arch. Méc. Appl., Gdańsk **1**, 229–254 (1949).

The author attempts to overcome the difficulties encountered when the classical concept of elastic instability is applied to practical problems. If the critical load of a column is defined as the axial load under which the deflection of the column will first equal the maximum allowable deflection, the Euler load represents the critical load under certain idealizing assumptions (initially straight column, centered load) which are never rigorously fulfilled in practice. Alternatively, the Euler load may be characterized as an upper bound of the critical load regardless of the initial curvature of the column or eccentricity of the load. If this point of view is adopted, however, the question arises of

how much this upper bound exceeds the practical value of the critical load. The author's discussion of this question is based on a definition of the critical load which, to the reviewer, appears to be rather arbitrary because it does not exhibit the necessary independence of the units in which load and deflection are measured. *W. Prager.*

Parkus, H. Beanspruchung und Schwingungen von Pleuelstangen. Österreich. Ing.-Arch. **3**, 222–235 (1949).

The author studies the deformations and stresses produced in a connecting rod due to time dependent axial and longitudinal loads. He reduces the problem to that of bending and stretching of a beam. By a Rayleigh-Ritz type of argument he reduces the two-dimensional problem (space and time) to a problem in which time is the only independent variable. The resultant differential equation is an equation of Hill type which he proceeds to discuss.

G. B. Thomas, Jr. (Cambridge, Mass.).

Lighthill, M. J., and Bradshaw, F. J. Thermal stresses in turbine blades. Philos. Mag. (7) **40**, 770–780 (1949).

The authors solve the thermo-elastic problem wherein a slab of material at one temperature is suddenly given a different surface temperature. A slab which has uniform thickness and one which is wedge-shaped are considered. The results are interpreted in terms of the stress system in a turbine blade when the hot gas flow is suddenly started or stopped. The discrepancy between the results and experimental evidence is very reasonable in view of the lack of information concerning certain features of the problem.

G. F. Carrier (Providence, R. I.).

Saharov, I. E. Bending of a rod under centrifugal force. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 329–330 (1949). (Russian)

Consider a homogeneous horizontal rod of constant cross section, rotating with constant angular velocity about a vertical axis to which one of the ends of the rod is clamped. If r is measured perpendicular to the axis, and $w(r)$ is the (small) deflection, then $Bw^{iv}-[T(r)w']'=f(r)$, where B is the constant flexural rigidity, $T(r)$ is the tension, and $f(r)$ is the transverse load, including the weight of the rod. The end conditions are $w(0)=w'(0)=w''(L)=w'''(L)=0$, where L is the length of the rod. Using $w'''(L)=0$, a second order nonhomogeneous ordinary differential equation is obtained for w', which is solved explicitly in terms of confluent hypergeometric functions (reference is made to C. Wells and R. Spence [J. Math. Physics **24**, 51–64 (1945); these Rev. **7**, 66]). The deflection $w(r)$ is then obtained by integration, and all three remaining arbitrary constants are determined to fulfill the other three end conditions. *J. B. Diaz.*

Il'in, Ya. K. Determination of stresses in a circular disk rotating about an eccentric axis. Doklady Akad. Nauk SSSR (N.S.) **67**, 803–806 (1949). (Russian)

This note contains an outline of a solution of the following problem. Determine the stresses in a thin elastic circular disk rotating with constant angular velocity about an axis normal to the plane of the disk, at a distance δ from its center. The problem is reduced to the solution of a plane problem of elasticity in the manner of Mushelišvili. The solution deduced by the author specializes to known results when $\delta=0$. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Sen Gupta, A. M. Stresses in some aeolotropic and isotropic circular disks of varying thickness rotating about the central axis. Bull. Calcutta Math. Soc. **41**, 129–139 (1949).

The stress distribution and deformation in a rotating cylindrically aeolotropic thin plate are determined. The thickness variations considered are of the form $r^{-\lambda}$ and $r^{\alpha}\exp(-\lambda r^{\beta})$, where r is the radial coordinate. The usual assumptions of thin plate theory are used. The author also reports the stress system associated with a rotating isotropic disc whose thickness variation is of the form $\exp(-\lambda r^{2})$.

G. F. Carrier (Providence, R. I.).

Sengupta, A. M. Thermal stresses in isotropic circular disks of varying thickness rotating about a central axis. Bull. Calcutta Math. Soc. **41**, 199–207 (1949).

An elastic disc rotating about its axis and subjected to a radial thermal gradient is considered. The stress distribution in such situations is found for the cases (1) thickness proportional to r^{λ}, temperature $=t_0+kr^{n}$; and (2) thickness proportional to $\exp(\lambda r^{q})$ and temperature proportional to $\log(b/r)$. The parameter r is the radial coordinate; all others are constants. *G. F. Carrier* (Providence, R. I.).

Morduchow, Morris. On internal damping of rotating beams. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1996**, 14 pp. (1949).

Consider an elastic beam rotating in a plane containing its "axis." The beam is oscillating with displacement normal to this plane. A certain type of damping is assumed and the decrease in amplitude per period of the free oscillations is investigated. It is concluded that the effect of rotation is to decrease this quantity. One should note that this decrease is associated primarily with the effect of the rotation on the period, but not with a change in amplitude decrease per unit time. A similar result is found for oscillations in the plane of rotation. *G. F. Carrier* (Providence, R. I.).

Pailloux, Henri. Sur la chute d'une barre. C. R. Acad. Sci. Paris **229**, 1118–1120 (1949).

Consider an elastic bar (in a vertical position) which is compressed under the influence of gravity. The author analyses the motion which ensues when the bar is suddenly allowed to fall freely. *G. F. Carrier* (Providence, R. I.).

Fu, C. Y. Energy flux of seismic surface waves. Acad. Sinica Science Record **2**, 322–327 (1949).

Il ne s'agit pas exactement d'ondes de surface au sens séismologique (ondes de Rayleigh ou de Love), mais d'ondes évanescentes au sens de la réflexion totale en optique: Dans la réflexion d'une onde plane transversale polarisée dans le plan d'incidence (onde SV) lorsque l'angle d'incidence dépasse l'angle critique, le flux d'énergie normal à la frontière contenu dans l'onde longitudinale (évanescente) est évidemment nul partout; au contraire il fluctue le long de la frontière si on remplace l'onde plane par un pinceau d'ondes cohérentes de directions voisines. *J. Coulomb* (Paris).

***Bartels, Julius. Mathematische Methoden der Geophysik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 7, pp. 89–99. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Smith, T. The contributions of Thomas Young to geometrical optics, and their application to present-day questions. Proc. Phys. Soc. Sect. B. **62**, 619–629 (1949).

The author describes Young's contributions to geometrical optics and the progress made in developing his ideas. Starting with the graphical construction of the refracted ray, a construction which has been rediscovered many times [the reviewer believes that it is really due to Huygens], the author discusses Young's formulae for the computation of astigmatism along a meridional ray, transforming them so as to give the final data with the help of matrix algebra as functions of the original data first for spherical surfaces and finally for aspherical surfaces which according to the author's contention will play an increasing role in the future development of optics. The problem of getting a system free of aberrations to the third order with the help of four mirrors leads to algebraic solutions which seem to have no direct practical value. The author's suggestion of using aspherical surfaces is based upon the fact that aspherical surfaces have a greater number of degrees of freedom than spherical surfaces. *M. Herzberger* (Rochester, N. Y.).

Gans, Ricardo. Simple deduction of the eiconal theorem. Revista Unión Mat. Argentina **14**, 3–15 (1949). (Spanish)

After writing down the Euler equations for the path of a light ray, the author specializes the paraxial rays in an axially symmetric system, and, from the solutions of the linearized equations, accounts for the terms of next higher order by a variation of the parameters.

A. S. Householder (Oak Ridge, Tenn.).

Wachendorf, F. Der Einfluss der achromatischen Bildfehler bis zur 5. Ordnung auf die Bildverundeutlichung auf Grund des quadratischen Mittlungsverfahrens. Optik **5**, 129–143 (1949).

The author considers the light distribution in the Gaussian image plane and applies the method of least squares to obtain a measure for the lack of sharpness, taking care of geometrical aberrations only. His discussion of the problem, which was first attacked by L. Schleiermacher [cf. Czapski and Eppenstein, Grundzüge der Theorie der optischen Instrumente, Leipzig, 1924, p. 721], shows clearly the strength and weakness of the method employed. To get results of practical interest, the methods will have to be transformed in such a way as to consider the influence of vignetting. *M. Herzberger* (Rochester, N. Y.).

Morais, C. Tolleranze dell'aberrazione sferica. Ottica (N.S.) **3**, 21–49 (1949).

The author proposes a special form of the wave surface as the best type of correction for the axis point of an optical system in the presence of aberration up to the seventh order. He also investigates the influence of spherochromatic aberrations. The paper is based on Rayleigh's quarter wavelength criterion. *M. Herzberger* (Rochester, N. Y.).

Polster, Harry D. Reflection from a multilayer filter. J. Opt. Soc. Amer. **39**, 1038–1043 (1949).

Glaser, Walter. Über die Theorie der elektrischen und magnetischen Ablenkung von Elektronenstrahlbündeln und ein ihr angepasstes Störungsverfahren. Ann. Physik (6) **4**, 389–408 (1949).

A derivation is given, based on Fermat's variational principle, of fundamental formulas of the author's earlier theory [Z. Physik **111**, 357–372 (1938)] of aberrations in electron-optical systems with two mutually perpendicular planes of symmetry through the axis. The criticism of Picht [same Ann. (5) **43**, 53–72 (1943); these Rev. **5**, 163] of this theory is shown to be unwarranted. The results of Picht and Himpan [same Ann. (5) **39**, 409–435, 436–477 (1941); these Rev. **4**, 289], so far as they are considered true, follow at once from the author's general expressions.

C. J. Bouwkamp (Eindhoven).

Sartori, Rinaldo. Uso dei vettori complessi per lo studio dei campi di vettori alternanti sinusoidali. Pont. Acad. Sci. Comment. **11**, 241–270 (1 plate) (1947).

In many electromagnetic problems it is convenient to represent the field vectors as the real and imaginary parts of a complex sinusoidal vector $(A_r+iA_c)e^{i\omega t}$, where A_r and A_c are three-dimensional vectors. As a development of this idea the author presents a systematic theory of complex vectors, $z=x+iy$, where x and y are vectors in a three-dimensional Cartesian space. These have many properties in common with ordinary complex scalars, but others peculiar to the vector nature of the components. For instance, the square of a complex vector is the scalar product of the vector by itself, $z^2=z\cdot z=x^2-y^2+2ix\cdot y$; and this may be a pure imaginary, when the vector is said to be symmetric, or it may be purely real, when the vector is called orthogonal. A circular vector is one which is both symmetric and orthogonal, that is, its square vanishes identically. Various general theorems on complex vectors are proved, somewhat analogous to the theorems of ordinary vector algebra, and the idea of a six-dimensional complex vector field is introduced.

In the second part of the paper the concepts of part 1 are applied to complex sinusoidal vectors. A generic form is $a=P\sin(\omega t+\varphi)+Q\cos(\omega t+\varphi)$, where P and Q are real vectors. Then for any given frequency the complex sinusoidal vector a is equivalent to the complex vector $A=(P+iQ)e^{i\varphi}$, and the properties of A may be used to develop the corresponding properties of a. *M. C. Gray.*

Debever, R. Électromagnétisme et géométrie. Bull. Soc. Math. Belgique **1** (1947–1948), 34–42 (1949).

Auluck, F. C., and Kothari, D. S. A note on the Riesz method and the method of residues. Proc. Roy. Soc. London. Ser. A. **198**, 170–173 (1949).

In classical electromagnetic theory, divergence-free results can be obtained by the λ-limiting process of Wentzel, Dirac and Pauli or by means of M. Riesz's generalisation of the Riemann-Liouville integral of fractional order [see N. E. Fremberg, Proc. Roy. Soc. London. Ser. A. **188**, 18–31 (1946); these Rev. **8**, 302]. The equivalence of the two methods was proved by S. T. Ma [Physical Rev. (2) **71**, 787–792 (1947); these Rev. **8**, 614]. The present paper demonstrates the equivalence of Riesz's method and a contour integral method developed by Frenkel [Lehrbuch der

Elektrodynamik, v. 1, Springer, Berlin, 1926, in particular, p. 127] for the case of a moving point charge.

E. T. Copson (Dundee).

Wheeler, John Archibald, and Feynman, Richard Phillips. Classical electrodynamics in terms of direct inter-particle action. Rev. Modern Physics **21**, 425–433 (1949).

A theory of direct inter-particle action giving a satisfactory account of the classical mechanical behavior of a system of interacting point charges and free of ambiguities associated with the self-field is developed. It is shown that this formulation is equivalent to a modified field theory. The physical significance of advanced action is dealt with in greater detail than has been done heretofore. The relativistic generalization of the Newtonian principle of action and reaction is obtained. For further discussions, see the papers by J. A. Wheeler and R. P. Feynman [same Rev. **17**, 157–181 (1945)] and R. P. Feynman [Physical Rev. (2) **74**, 1430–1438 (1948); these Rev. **10**, 345].

C. Kikuchi (East Lansing, Mich.).

Müller, Claus. Die Grundzüge einer mathematischen Theorie elektromagnetischer Schwingungen. Arch. Math. **1**, 296–302 (1949).

The author discusses the existence and uniqueness of solutions of Maxwell's equations, which he writes in the form

$$\operatorname{curl} H + i\omega\epsilon E = j, \quad \operatorname{curl} E - i\omega\mu H = -j',$$

where ϵ and μ are complex point functions and it is assumed that the electric and magnetic current densities are differentiable functions which differ from zero only inside a finite closed region V. Four cases are considered: (1) ϵ and μ are constant everywhere; (2) ϵ and μ are differentiable functions and grad $\epsilon=0$, grad $\mu=0$ outside V; (3) ϵ and μ have constant values both inside and outside V but these values change abruptly in crossing the boundary surface F of V; (4) same as (3) except that $\epsilon=\infty$ inside V.

M. C. Gray (Murray Hill, N. J.).

***Watson, W. H. Discontinuity in electromagnetism.** Proc. Symposia Appl. Math. **2**, 49–57 (1950). $3.00.

The present paper summarizes the author's earlier investigations on the indicated subject. See W. H. Watson [Canadian J. Research. Sect. A. **23**, 33–38, 39–46 (1945); these Rev. **6**, 221]. *C. Kikuchi* (East Lansing, Mich.).

Bureau, Florent. La notion d'espace et la mécanique de l'électron. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 542–558 (1949).

The author shows that one may associate with a general second order linear partial differential equation, which is invariant under transformations of coordinates, a metric tensor a_{ij} of a Riemannian space (the coefficients of the second derivative terms), a vector φ_i (the coefficients of the first derivatives), and a scalar h (the coefficient of the unknown function). If the equations of the bicharacteristics are suitably modified they may be interpreted as giving the equations of motion of a system of mass h moving in the gravitational field a_{ij} and the electromagnetic field with vector potential φ_i. The same result is said to hold for a system of equations of hyperbolic type. No interpretation (physical or mathematical) is given to the modification of the equations describing the bicharacteristics.

A. H. Taub (Urbana, Ill.).

***Taub, A. H. Orbits of charged particles in constant fields.** Proc. Symposia Appl. Math. **2**, 82 (1950). $3.00

Abstract of a paper which appeared in Physical Rev. (2) **73**, 786–798 (1948); these Rev. **10**, 223.

***Kron, A.-W. Anwendung der Mathematik in der Elektrotechnik.** Naturforschung und Medizin in Deutschland 1939–1946, Band 7, pp. 63–79. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Bibliography of 149 papers published during the indicated period. *C. J. Bouwkamp* (Eindhoven).

Price, A. T. The induction of electric currents in nonuniform thin sheets and shells. Quart. J. Mech. Appl. Math. **2**, 283–310 (1949).

General equations are obtained for the induction of electric currents in a nonuniform thin conducting sheet by periodic and aperiodic magnetic fields. It is assumed that the variations of the magnetic field are sufficiently slow to neglect the displacement current. These equations are solved by infinite series for nonuniform plane sheets and spherical shells. Numerical examples illustrate some important features. *E. Kogbetliantz* (New York, N. Y.).

Toraldo di Francia, G. Sur les lois générales de la diffraction. Rapport critique. Rev. Optique **28**, 597–611 (1949).

This is a survey article treating various aspects of the theory of diffraction, with emphasis on optics. A bibliography of 52 items is included. *C. J. Bouwkamp.*

Maue, A.-W. Zur Formulierung eines allgemeinen Beugungsproblems durch eine Integralgleichung. Z. Physik **126**, 601–618 (1949).

The author discusses how scalar as well as electromagnetic diffraction problems may be solved with the aid of integral equations, set up for the boundary values of the wave functions over the diffracting obstacle. For instance, the problem of electromagnetic diffraction by a perfect conductor is equivalent to that of solving a certain integral equation for the electric surface-current density. The general theory is illustrated by the diffraction of a plane wave at an infinite circular cylinder. In addition, there is a thorough discussion of the character of singularities at sharp edges. *C. J. Bouwkamp* (Eindhoven).

Müller, Cl. Das allgemeine Beugungsproblem und die Separation der Maxwellschen Gleichungen nach Bromwich. Abh. Math. Sem. Univ. Hamburg **16**, 95–103 (1949).

It is shown that the method of Bromwich [Philos. Mag. (6) **38**, 143–164 (1919)] for solving Maxwell's equations is effective only for systems of either spherical or cylindrical symmetry. The author suggests that the general diffraction problem should be tackled independently of the geometry of the obstacle. The relevant theory is to be developed in a series of papers to be published in the future.

C. J. Bouwkamp (Eindhoven).

Papas, Charles H. On the infinitely long cylindrical antenna. J. Appl. Phys. **20**, 437–440 (1949).

The author determines the distant field of an infinitely long perfectly conducting cylindrical antenna of radius a, driven at some cross-section by an electromotive force V, circumscribing the cylinder in a narrow peripheral band. Then the nonvanishing field components are E_ρ, E_z, H_φ and

it is known that E_s may be expressed as an infinite integral involving Hankel functions. At great distances from the cylinder the Hankel function may be replaced by its asymptotic expansion, and the integral for E_s is then evaluated by the method of steepest descents. The final formulas are

$$E_s(r,\theta)=-\frac{Vi}{\pi}\frac{e^{ikr}}{r}\frac{1}{H_0^{(1)}(ka\cos\theta)},\quad E_\theta(r,\theta)=\frac{E_s(r,\theta)}{\cos\theta},$$

$$H_\varphi(r,\theta)=\frac{E_\theta(r,\theta)}{120\pi}.$$

A simple formula for the radiation conductance at low frequencies is obtained by integrating the radial component of the Poynting vector over a large sphere, and an asymptotic form for the current on the cylinder at large distances from the localized emf is also derived. *M. C. Gray.*

Papas, Charles H. Diffraction by a cylindrical obstacle. Tech. Rep. no. **83**, Cruft Laboratory, Harvard University, Cambridge, Mass., 1949. i+23 pp. (2 plates)

The author applies Levine and Schwinger's [cf. Physical Rev. (2) **74**, 958–974 (1948); **75**, 1423–1432 (1949); these Rev. **10**, 221, 764] variational principle to obtain an approximate solution, for the short-wave range, to the diffraction of a plane electromagnetic wave by an infinitely long, perfectly conducting cylinder, on the assumption that the incident wave is polarized in a direction parallel to the axis of the cylinder. The scattering cross-section is found to be $4a(1-4\lambda^2/\pi^2a^2)^{-1}$ as $\lambda\to 0$, where a is the radius of the cylinder and λ the wavelength. The low-frequency scattering cross-section is calculated in the conventional manner. The resulting curves overlap at $2\pi a/\lambda\approx 10$.

C. J. Bouwkamp (Eindhoven).

Vasseur, Jean-Pierre. Diffraction des ondes électromagnétiques par un écran plan parfaitement conducteur. C. R. Acad. Sci. Paris **229**, 179–181 (1949).

Vasseur, Jean-Pierre. Nouvelle solution du problème de la diffraction des ondes électromagnétiques par un écran plan parfaitement conducteur. C. R. Acad. Sci. Paris **229**, 586–587 (1949).

In the first paper, the author recalls that Copson [Proc. Roy. Soc. London. Ser. A. **186**, 100–118 (1946); these Rev. **8**, 179] omitted a certain line integral in his field expressions. Whereas this integral actually vanishes in Copson's solution for the small circular aperture, the author suggests that in general it may not vanish. Thus including a line integral, the author formulates the solution of the diffraction problem in terms of integral equations for the fictitious magnetic currents in the aperture. No attempt is made to solve these equations. In the second paper, an alternative procedure is indicated in which electric currents in the screen now play the part of the magnetic currents in the aperture in the former case. Comparison of the two solutions leads to the rigorous principle of Babinet, as formulated by Booker and Copson. The author claims that "la démonstration donnée ici est rigoureuse." The reviewer is of a different opinion, however. First, the solution to the integral equations is certainly not unique without stating definite boundary conditions at the rim of the screen. Second, it seems plausible, in view of the known solutions of Sommerfeld (half-plane) and Bethe and Copson (small circular hole) that the component of the current normal to the rim vanishes at the rim, so that Vasseur's line integral is zero. In this respect the reviewer agrees with Copson's remark [see footnote reference 24 in a paper by Miles, J. Appl. Phys. **20**, 760–771 (1949); these Rev. **11**, 141].

C. J. Bouwkamp (Eindhoven).

Ljunggren, T. Contributions to the theory of diffraction of electromagnetic waves by spherical particles. Ark. Mat. Astr. Fys. **36A**, no. 14, 36 pp. (1949).

This paper is devoted to the problem of diffraction of a monochromatic plane-polarized electromagnetic wave (wave number k) by a homogeneous sphere (radius a). The author starts with the classical series of Mie and Debye for the electric and magnetic components of the secondary field, involving first-order Legendre polynomials and spherical Bessel functions. These series are not suitable for numerical purposes unless $ka\lesssim 10$, as is evident from a number of papers cited by the author. For greater values of ka they have to be transformed in one way or another. Earlier work in this connection is briefly mentioned. The author's method is based on a paper by Waller [thesis, Uppsala, 1925]. It is claimed that the theory here presented offers an essential simplification as compared to the theory of van der Pol and Bremmer [Philos. Mag. (7) **24**, 825–864 (1937)]. The reviewers are of a different opinion, however.

The coefficients (A_l, B_l) of the series for the scalar wave potentials (expansions in terms of Legendre functions) are developed into certain geometrical series, equation (15). [The reviewers cannot see any difference between van der Pol and Bremmer's (71) and the author's (15), apart from a change in notation, which is partly due to the difference in the choice of the complex time factor.] The novelty in the author's treatment is that the series in Legendre functions for calculating the secondary field are broken off after the $(ka-1)$th term, where ka is for simplicity taken equal to an integer. This approximation is permissible for $ka\gg 1$; the author suggests that the relevant expressions are also useful for extrapolation down to $ka=10$, say, which question will be considered in detail in a forthcoming paper by the author. In summing the first $(ka-1)$ terms, the author applies certain saddle-point approximations for the Hankel functions [tangent approximation; see, e.g., the book reviewed below, chapter 3, section 2].

The secondary field in the wave zone is split up into four parts: (1) the diffraction term, (2) a term due to external reflection at the sphere, (3) a refraction term, and (4) an infinite series of so-called rainbow terms. [Again, this splitting is essentially equivalent to that given by van der Pol and Bremmer, except that these authors add part (1) to the primary field, which is of importance in the residue method, not considered in the paper under review; furthermore, they consider part (3) as the first rainbow term.] In the author's approximation, omitting terms of order exceeding $(ka-1)$, part (1) can be rigorously summed. On application of certain saddle-point approximations for the functions involved, agreement is obtained with results predicted by elementary diffraction theory. As to the remaining three parts, the sum over the $(ka-1)$ Legendre functions is transformed by means of Poisson's formula, leading for each part to a series of integrals. [Van der Pol and Bremmer used Watson's transformation for the same purpose; there is no essential difference between these two transformations, except that van der Pol and Bremmer kept to the infinite series rather than break them off at the kath term. There consequently exists no difference as to the derivation of the geometric-optical interpretations for the different parts with the aid of the saddle-point method.] The author is correct in stating

that equations (111) and (112) of van der Pol and Bremmer's paper are incorrect. [A factor exp ($il\pi$) is to be added in the author's equation (61).]

C. J. Bouwkamp and H. Bremmer (Eindhoven).

***Bremmer, H. Terrestrial Radio Waves. Theory of Propagation.** Elsevier Publishing Company, Inc., New York, N. Y., Amsterdam, London, Brussels, 1949. x+343 pp. (4 plates). $5.50.

This book aims at the discussion of the mathematical-physical methods for the computation of transmitter fields, and it carries on the investigations started by B. van der Pol and the author. Basically, the problem is that of propagation of electromagnetic waves around a sphere (the earth) and it is worth noting that although investigations of this problem are numerous (as is to be expected in view of the important practical applications of the results), yet this would seem to be the first book devoted entirely to a detailed mathematical discussion carried through to computational details. The book was largely written while its author was cut off from information on much of the work in this field carried out simultaneously in many countries. Under the circumstances it is inevitable that the account presented here should be far from complete, and this in spite of the author's endeavour to indicate some of the progress achieved in other countries.

The work consists of an introduction (chapter I) and two parts (of five chapters each) dealing with the theory for a homogeneous and inhomogeneous atmosphere, respectively. In chapter II the field of an electric or magnetic dipole situated at an arbitrary height above the surface of the earth is represented by the well-known series of Bessel functions and Legendre functions. In this chapter there is also a section on the application of the saddle-point method to waves, showing that asymytotic representations of wave functions correspond to geometric-optical approximations to wave motion. Chapter III is in a sense the crucial chapter. In it the author develops a modified form of G. N. Watson's transformation of the infinite series into a complex integral which is the basis of all further analysis. In evaluating the complex integral as a series of residues (as on all similar occasions) the physical meaning of each term is carefully discussed. In this chapter there are also the required asymptotic representations of Bessel and Legendre functions; the position of the poles of the integrand of the modified Watson integral is discussed by means of these asymptotic representations. In chapter IV the series of residues is used for carrying through the computations. More detailed results are given for the case of a transmitter situated directly on the surface of the earth. In chapter V the geometric-optical approximation is derived from the modified Watson integral by means of the saddle-point method, and the results are used for instance for a numerical discussion of the reflection coefficient as a function of the frequency and soil conditions. Chapter VI brings a survey of the formulae required for the numerical computation of the electromagnetic field, together with a wealth of numerical results given in a large number of diagrams.

In chapter VII the author takes up the discussion of propagation in a nonhomogeneous atmosphere. The parameters are at first any functions of the distance from the centre of the earth (spherical layers), but in the later parts of this chapter the author follows Eckersley in assuming that the ratio of the wave numbers in the troposphere and earth is expressed in terms of the distance from the centre of the earth, r, in the form $(1-b+c(a/r)^2)^{\frac{1}{2}}$, where b and c are (nearly equal) constants and a is the radius of the earth. The advantage of this set-up is that it leads to infinite series of Bessel functions and Legendre functions (like the series in the homogeneous case), and yet represents the refraction in the lower part of the atmosphere to a sufficient degree of approximation. In chapter VIII the series of Bessel and Legendre functions is transformed into a complex integral which in its turn can be evaluated as a series of residues, and also discussed by means of the saddle-point method. In this chapter the lower part of the atmosphere is assumed to be homogeneous, but any variation of the wave number (as function of r) is allowed in the higher layers. In chapter IX propagation through a nonhomogeneous atmosphere is further investigated with special reference to the relation between the geometric-optical series on the one hand, and the residue series on the other hand. Chapter X brings special applications, propagation of long waves, short waves (absorption being neglected), effect of the absorption on short wave fields, super-refraction, and radio waves from outside the earth. Throughout the discussion the magnetic field of the earth has been ignored, and the final chapter XI contains a discussion of the effect of the magnetic field of the earth.

It is a very difficult task to write a book in which involved mathematical analysis and numerical computations enter side by side, since it is inevitable that some readers should wish to study the mathematical work without being interested in the numerical computations, while a larger group of readers will want to use the numerical results, or compute by means of the formulae provided, without having the inclination, or indeed the necessary preparation, to wade through the mathematical analysis. By arranging his mathematical work so that almost every step can be interpreted in terms of physics, and by segregating numerical computations as much as possible, the author has succeeded to a marked degree in his difficult task. *A. Erdélyi.*

***Johnston, Dave, Jr. Correlation of radiation patterns to circularly symmetric aperture distributions.** Bumblebee Series, Rep. no. 103. The Johns Hopkins University Applied Physics Laboratory, Silver Spring, Md., 1949. vi+59 pp.

The author is concerned with the determination of the amplitude distribution of electric and magnetic field vectors over a circular aperture if the distant field is prescribed. This postulated distant field is understood to be of such a nature that it may be set up by circularly symmetric plane polarized waves over the illuminated aperture. Therefore the author's problem is essentially a scalar one, as encountered in acoustics in connection with a vibrating piston in an infinite baffle. The results of the paper are well known, though the author seems not to be aware of this fact. Moreover, the physicist will be puzzled by the statement that "the solution contains the small, generally neglected, component of energy flow normal to the direction of propagation." In fact, the author's argument that the line integrals in equations (2.01) and (2.13) are zero does not hold. Their omission leads to the false result cited, according to which the distant field would not be transverse. However, this does not affect the author's numerical calculations, which are given in great detail. His result that certain patterns are not realizable in theory lacks proof, and is false in the sense that any "smooth" pattern can be approximated to any degree of accuracy. *C. J. Bouwkamp.*

Ginzburg, V. L. On the theory of the propagation of electromagnetic waves in a magneto-active medium. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **18**, 487–500 (1948). (Russian)

The present paper, which is a continuation of a previous work by the author [Acad. Sci. USSR. J. Phys. **7**, 289–304 (1943); these Rev. **6**, 54], develops further the theory of electromagnetic waves in an inhomogeneous magneto-active medium (the ionosphere under the influence of the earth's magnetic field). An approximate solution of the propagation equations is obtained in the region of the layer where the wave undergoes reflection and at the beginning of the layer, where geometrical optics is not valid. By means of an approximation method the author treats further the phenomenon of trebling, the occurrence of reflection at three positions, for small angles between the magnetic field and the direction of propagation. By the use of kinetic theory considerations he derives an expression for the effective collision frequency in the presence of a magnetic field.

N. Rosen (Chapel Hill, N. C.).

***Buchholz, H. Elektromagnetische Wellen in Hohlleitern und verwandte Probleme.** Naturforschung und Medizin in Deutschland 1939–1946, Band 7, pp. 81–87. Dieterich'sche Verlagsbuchhandlung, Wiesbaden, 1948. DM 10 = $2.40.

Abstracts of seventeen theoretical papers [some of which have not yet been published] by Buchholz, Knaack, Magnus, and Oberhettinger. *C. J. Bouwkamp* (Eindhoven).

Buchholz, Herbert. Der Einfluss von Trennfugen auf das Hohlleiterfeld. Arch. Elektr. Übertragung **2**, 14–22 (1948).

The author investigates how a narrow slot, cut in the wall of a circular cylinder in the transverse direction, affects the propagation of waves inside the cylinder. The field of the discontinuity presented by the slot is equivalent to that of a ring of axially directed magnetic dipoles, the amplitudes of which are calculated. The theory is illustrated by some numerical examples. *C. J. Bouwkamp* (Eindhoven).

Ledinegg, Ernst. Über Rand- und Sprungwertprobleme der Maxwellschen Gleichungen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **156**, 417–440 (1948).

The extension of Huygens' principle to electromagnetic theory has been established by Larmor, Kottler, and others [cf. B. B. Baker and E. T. Copson, The Mathematical Theory of Huygens' Principle, Oxford University Press, 1939; these Rev. **1**, 315]. The author's solution of this problem is novel in that no appeal is made to physics. The results are applied to cavity resonators with perfect walls. Maxwell's equations are then shown to be equivalent to two linear integral equations with symmetric kernels. The existence of real eigenvalues and eigenfunctions of arbitrary cavities is discussed. *C. J. Bouwkamp* (Eindhoven).

Ledinegg, E. Zur Störungsrechnung elektromagnetischer Hohlräume mit einigen Anwendungen auf die experimentelle Bestimmung der Dielektrizitätskonstanten im Zentimeter-Wellengebiet. Österreich. Ing.-Arch. **3**, 128–139 (1949).

A new derivation is given of an expression due to Müller [Hochfrequenztech. Elektroak. **54**, 157–161 (1939); these Rev. **1**, 275] for the resonance-frequency shift of a cavity resonator as resulting from small changes in the dielectric and magnetic constants of the medium inside the resonator, or from a small deformation of the wall, or from both. The author's derivation shows that Müller's formula is more widely applicable than would be inferred from Müller's original calculations. This is checked against an example that has been solved rigorously [E. Ledinegg, Naturwissenschaften **31**, 569–570 (1943)]. Some experimental details are included. *C. J. Bouwkamp* (Eindhoven).

Ledinegg, E. Störungsrechnung elektromagnetischer Hohlräume bei mehrfachem Eigenwert. Österreich. Ing.-Arch. **3**, 215–221 (1949).

The author investigates the effect of small deformations of the wall of a cavity resonator on the latter's multiple resonance frequencies. Explicit expressions are given for the case of two-fold degeneracy. *C. J. Bouwkamp.*

***Stevenson, A. F. Wave propagation in electromagnetic horns.** Proc. Symposia Appl. Math. **2**, 84 (1950). $3.00.

Lewin, L. Electromagnetic radiation from wave-guides and horns. Nature **164**, 311 (1949).

The author announces that in light of a recent paper of Levine and Schwinger [Physical Rev. (2) **73**, 383–406 (1948); these Rev. **9**, 393], he can find the radiation properties of a pair of semi-infinite parallel plates for the E and H plane polarization and gives the reflection coefficients (magnitude and phase) for these cases. The details are promised elsewhere. [The H plane case was published by the reviewer, Quart. Appl. Math. **6**, 157–166, 215–220 (1948); these Rev. **10**, 89, 222. Simultaneously, L. A. Vaĭnšteĭn published both the E and H plane cases, Izvestiya Akad. Nauk SSSR. Ser. Fiz. **12**, 144–165, 166–180 (1948); these Rev. **10**, 659.]

A. E. Heins (Pittsburgh, Pa.).

***Rice, S. O. Reflections from bends and corners in electromagnetic wave guides.** Proc. Symposia Appl. Math. **2**, 83 (1950). $3.00.

Abstract of a paper which appeared in Bell System Tech. J. **28**, 104–135, 136–156 (1949); these Rev. **10**, 660.

Piloty, Hans. Die Halbierung eines symmetrischen Reaktanz-Vierpols. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 187–236 (1949).

By "halving" a symmetric reactance 4-pole is meant finding a new 4-pole which when put in series with its mirror image yields a 4-pole equivalent to the original 4-pole. The author establishes necessary and sufficient conditions for a symmetric reactance 4-pole to be "halved" and shows that not every symmetric reactance 4-pole satisfies these conditions. *N. Levinson* (Cambridge, Mass.).

Suezaki, Teruo. Theory of locking phenomena in nonlinear oscillation. Proc. Fac. Engrg. Keiogijuku Univ. **1**, 59–67 (1948). (Japanese. English summary)

In the presence of nonlinear circuit elements, the plate current can generally be expressed in the form

$$i = \sum_{n=1}^{\infty} g_n (V_g + V/\mu)^n,$$

where μ is the vacuum tube amplification constant, V_g the grid voltage, V the plate voltage, and g_1 the mutual conductance. It is shown in the present paper that the essential features of locking phenomena can be predicted by taking the plate current to be a cubic function of the voltage, namely, $i = g_1(V_g + V/\mu) + g_3(V_g + V/\mu)^3$, and neglecting all higher powers. *C. Kikuchi* (East Lansing, Mich.).

Brazma, N. A. Solution of the fundamental problem of the propagation of electromagnetic processes in a multiconductor system. Doklady Akad. Nauk SSSR (N.S.) **69**, 313–316 (1949). (Russian)

The matrix telegraphic equation

$$u_{xx}=LCu_{tt}+(LG+RC)u_t+RGu$$

($u(x, t)$ a column matrix of voltages, L, C, R, G square matrices of inductances, capacitances, resistances and leakances), with initial and boundary conditions $u(x, 0)=0$, $u_t'(x, 0)=0$, $u(0, t)=u_0=$constant, $u(l, t)=0$, is solved by a series in terms of normal functions $a(\frac{r}{l})\exp\{\lambda(\frac{r}{l})t\}\sin(k\pi x/l)$, together with a steady-state term,

$$u_0\sinh\{\sqrt{(RG)}(l-x)\}/\sinh\{\sqrt{(RG)}l\},$$

the latter term needing a suitable interpretation. Full details are given for the case of two lines. An algebraic criterion is cited for the propagation constants $\lambda(\frac{r}{l})$ to have negative real parts; the latter is shown to be the case if the mutual parameters are sufficiently small, i.e., if the system approximates to a set of isolated conductors. Coincident modes are shown to correspond to resonance. *F. V. Atkinson.*

***Montgomery, C. G., Dicke, R. H., and Purcell, E. M., editors. Principles of Microwave Circuits.** McGraw-Hill Book Company, Inc., New York, N. Y., 1948. xvi+486 pp. $6.00.

The reduction of a solution of a boundary value problem in microwaves to a point where it can be useful to the engineer or experimental worker requires much analysis. In the large, it is the goal of this book to fill in the gap between low frequency concepts of lumped circuit phenomena and the field of microwaves. The boundary value problems are not discussed in this volume, but free use is made of such results. The book opens with an introductory chapter indicating the scope of the volume as well as the physical assumptions. This is followed by a summary of the Maxwell theory and certain wave solutions. Applications are given to parallel plate regions, wave guides and cavities. Some discussion of the wave guide as a transmission line and linear network theory follows.

Having discussed these preliminaries, the authors embark upon the main part of their program. First we find a discussion of general microwave circuits theorems. This is followed by a section on wave guide circuit elements. The electrical properties of some of these elements have been computed theoretically; in other cases only experimental results are available. The remainder of the volume is devoted to a study of the circuitry which arises from various geometrical structures which are encountered in microwave theory.

From the mathematical standpoint, the last chapter is of interest. By taking advantage of the symmetries in many of these microwave structures, it is possible to decompose a fairly complex problem into a series of more simple ones. This is accomplished by introducing some matrix algebra and group theory concepts. The results are too detailed to describe here. *A. E. Heins* (Pittsburgh, Pa.).

Zimmermann, F. Die Auflösung elektrischer Netze mittels Matrizen. Österreich. Ing.-Arch. **3**, 140–180 (1949).

The theory of single and simultaneous faults on three-phase transmission systems treated in a previous publication by tensorial methods is here reproduced with slight variations. The principal differences are mainly in terminology and notation. *G. Kron* (Schenectady, N. Y.).

Quantum Mechanics

Shôno, Naomi, and Oda, Nobuo. On the Dirac's indefinite metric space. Progress Theoret. Physics **4**, 358–363 (1949).

The authors give a relatively explicit construction of the Dirac indefinite-metric formalism [cf. Pauli, Rev. Modern Physics **15**, 175–207 (1943); these Rev. **5**, 277]. If P is a projection on a Hilbert space and $\eta=I-2P$ [incorrectly printed as $I-P$ but corrected on p. 581], the observables are the operators which are self-adjoint relative to the form $\langle x, y\rangle=(\eta x, y)$, where $(\cdot,\cdot)$ is the original inner product. The harmonic oscillator is treated, the results obtained being essentially contained in the work of Dirac and Pauli.

I. E. Segal (Chicago, Ill.).

Møller, C. On the definition of the centre of gravity of an arbitrary closed system in the theory of relativity. Communications Dublin Inst. Advanced Studies. Ser. A. no. **5**, 42 pp. (1949).

In classical theory a symmetrical energy-momentum tensor is used to define, for an arbitrary closed dynamical system, a centre of gravity in the rest-reference-system (total linear momentum zero) which is fixed in that reference-system S^0, and moves with uniform velocity in an arbitrary inertial reference-system S. Each reference-system S has its centre of mass: these centres of mass coincide only if the angular-momentum tensor relative to the centre of gravity vanishes. A classical system with positive energy density and given inner angular momentum and rest mass must have a nonzero extension in the centre of gravity reference-system. The motions of a classical spinning particle found by Mathisson [Acta Physica Polon. **6**, 163–200, 218–227 (1937)] are motions of pseudo-centres of gravity.

In quantum theory a postulated symmetrical energy-momentum tensor defines q-numbers representing the total momentum four-vector and the total angular-momentum tensor, which in turn define transformations of field variables corresponding to an infinitesimal displacement of space-time origin and to an infinitesimal homogeneous Lorentz transformation. One form of the transformation laws leads to a reasonable quantum-mechanical definition of the coordinates of the centre of gravity but these do not commute and there is a corresponding uncertainty in the determination of its position. A point whose coordinates commute may be defined on the line joining the centre of gravity and the centre of mass for a particular S, different for different S. The scheme is applied to the Dirac electron theory.

C. Strachan (Aberdeen).

Bodiou, Georges. Sur la caractère problématique de certaines disjonctions a posteriori susceptible de fonder la composition ondulatoire des probabilités (Introduction de l'espace de Hilbert-Hermite). C. R. Acad. Sci. Paris **230**, 180–182 (1950).

Wigner, E. P. Relativistische Wellengleichungen. Z. Physik **124**, 665–684 (1948).

To treat the relativistic one-particle problem in quantum mechanics one usually proceeds by setting up linear equations which are relativistically invariant; the solutions of the equations then form a relativistically invariant manifold describing the possible states of the system. An alternative approach, and the one used in this paper, is to determine the invariant linear manifold directly and then to find the wave equation as an equation that can characterize the manifold. In the case of a particle having a finite rest-mass

this point of view leads only to the manifolds previously obtained by E. Majorana [Nuovo Cimento (N.S.) **9**, 335–344 (1932)] by the first procedure. However, in the case of a particle of zero rest-mass the manifolds cannot be obtained as limits of those for a finite rest-mass. For a given value of momentum they contain only one, two, or an infinite number of polarization states. A set of equations which characterize a manifold of the latter type in coordinate space is given by

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\sum\frac{\partial^2}{\partial x_i^2}\right)\Phi=0,$$

$$\left(\tau\frac{\partial}{\partial t}+\sum\xi_i\frac{\partial}{\partial x_i}\right)\Phi=0,$$

$$(\sum\xi_i^2-c^2\tau^2-I^2)\Phi=0,$$

$$\left(\sum\frac{\partial^2}{\partial x_i\partial\xi_i}-\frac{1}{c^2}\frac{\partial^2}{\partial t\partial\tau}-\Xi\right)\Phi=0,$$

where x_1, x_2, x_3 are Cartesian coordinates, t is the time, c the velocity of light, ξ_1, ξ_2, ξ_3 and $\xi_4=c\tau$ are the components of a four-vector, Ξ and I^2 are real constants, and the summations are taken over $i=1, 2, 3$. By a change of scale of ξ, τ one can make $I^2=1$. The scalar product of two wave functions is then given by

$$(\Phi_1, \Phi_2)=\tfrac{1}{2}\int\left(\frac{1}{c^2}\frac{\partial\Phi_1^*}{\partial t}\frac{\partial\Phi_2}{\partial t}+\nabla\Phi_1^*\cdot\nabla\Phi_2\right)\delta(x\cdot\xi)\delta(\xi^2-1)dxd\xi,$$

where δ is the Dirac delta-function, $x\cdot\xi=\sum x_i\xi_i$, $\xi^2=\sum\xi_i^2$ and $dxd\xi=dx_1dx_2dx_3d\xi_1d\xi_2d\xi_3$. *N. Rosen.*

Kallmann, H., und Päsler, M. Wellenmechanische Störungsrechnung im "Unterbereich" der Laplace-Transformation. (Mit einer Anwendung auf den Stark-Effekt.) Z. Physik **126**, 734–748 (1949).

Bothe, W. Zur Theorie der Bremsung von Neutronen. Z. Physik **125**, 210–224 (1948).

It is known that when neutrons (of mass 1) are scattered elastically (and isotropically in the centre of gravity system) by nuclei of mass M, the probability that after a single collision $x=\log(E_0/E)$ (where E_0 is the initial energy of the neutron) lies between ξ and $\xi+d\xi$ is given by $p_1(\xi)=e^{-\xi}/(1-\beta)$ for $\xi<l=-\log\beta$ and zero otherwise. The author shows how the various problems in the theory of the stationary distribution of neutrons in an infinite medium can be solved in terms of this probability $p_1(\xi)$. Thus the distribution of energy after n collisions is given by

$$p_n(x)=(2\pi)^{-1}\int_{-\infty}^{\infty}e^{i\alpha x}\left(\int_0^l p_1(\xi)e^{-i\alpha\xi}d\xi\right)^n d\alpha$$

$$=(2\pi)^{-1}(1-\beta)^{-n}\int_{-\infty}^{\infty}e^{i\alpha x}[(1-e^{-(i\alpha+1)l})/(i\alpha+1)]^n d\alpha.$$

The evaluation of the integral on the right-hand side by the method of residues gives

$$p_n(x)=ne^{-x}(1-\beta)^{-n}\sum_{\mu=0}^{\nu}{}'(-1)^\mu(x-\mu l)^{n-1}/\mu!(n-\mu)!,$$

$$\nu l<x<(\nu+1)l.$$

In terms of $p_n(x)$ the probability $P(x)$ that a neutron will have attained at one time or another a value of x in the $(\nu+1)$th interval is given by $P(x)=\sum_{n=\nu+1}^{\infty}p_n(x)$. An evaluation of this sum gives

$$P(x)=\frac{e^{-x}}{1-\beta}\sum_{\mu=0}^{\nu}{}'\frac{(-1)^\mu}{\mu!}(y_\mu+\mu)y_\mu^{\mu-1}e^{y_\mu},$$

where $y_\mu=(x-\mu l)/(1-\beta)$. Other related problems are also considered. *S. Chandrasekhar* (Williams Bay, Wis.).

***Feenberg, Eugene. A method of analytical continuation in the eigenvalue and scattering problems of quantum theory.** Proc. Symposia Appl. Math. **2**, 20 (1950). $3.00.

Abstract of a paper which appeared in Physical Rev. (2) **74**, 206–208 (1948); these Rev. **10**, 154.

Heitler, W., and Ma, S. T. Quantum theory of radiation damping for discrete states. Proc. Roy. Irish Acad. Sect. A. **52**, 109–125 (1949).

The quantum theory of radiation damping [W. Heitler, The Quantum Theory of Radiation, 2d ed., Oxford, 1944] is used as the basis for a treatment of transition-processes of atoms between discrete states with emission and absorption of radiation. It is shown that by working in an "energy representation," in which the energy of the system is a diagonal operator, a general problem of time-dependent perturbation theory can be reduced to a time-independent problem, the solution of which is simpler and shorter. This procedure leads to a general description of damping processes, which includes as special cases (i) the Weisskopf-Wigner theory of spectral line-breadths, and (ii) the Heitler theory of collisions between free particles. Two problems are worked out as examples of the method, (i) resonance fluorescence, and (ii) the transfer of excitation by means of a quantum passing from one atom to a similar atom at a large distance. In the latter example, it appears that the theory gives correctly the velocity of propagation of the light between the two atoms, contrary to the view expressed by Ferretti and Peierls [Nature **160**, 531–532 (1947)]. *F. J. Dyson* (Birmingham).

Bennett, J. G., Brown, R. L., and Thring, M. W. Unified field theory in a curvature-free five-dimensional manifold. Proc. Roy. Soc. London. Ser. A. **198**, 39–61 (1949).

The authors postulate a five-dimensional flat space as an "absolute-world" in which physical phenomena are represented. An idealised physical observer and his measuring system of rigid rulers and clocks correspond to a set of five unit vectors, L_i $(i=1, \cdots, 5)$ at each point of the five-dimensional manifold. [The consistency of the notion of a rigid body with finite maximum signal velocity is never raised in the paper.] The nature of this quintuplet of vector-fields is assumed to vary with the physical fields acting on test bodies but no method is given for determining this variation in terms of the matter and energy present. However, a first order theory is presented in which the deviation of these vector fields from the coordinate directions in the absolute world is assumed to be small. It is then shown that if the motion of a test body is a straight line in the "absolute-world" it would be "observed" to move on a curved path when referred to the L_μ $(\mu=1, 2, 3, 4)$. In the argument an additional vector field ω orthogonal to the L_μ is used.

It is claimed that when a massive uncharged body δM is rigidly attached to a mass M, δM produces a magnetic field as if it had a charge numerically equal to its gravitational mass but without the corresponding electric field.

This is ascribed to the fact that the rigid attachment of δM to M produces a set of vectors L_i which differ from the L_i' when no constraint is present. However, it is assumed that ω and ω' are the same. In this manner the authors "derive" Blackett's relation between magnetic dipole moment and angular momentum. *A. H. Taub* (Urbana, Ill.).

Bergmann, Peter G., and Brunings, Johanna H. M. Nonlinear field theories. II. Canonical equations and quantization. Rev. Modern Physics **21**, 480–487 (1949).

The general covariant field theory introduced earlier [Physical Rev. (2) **75**, 680–685 (1949); these Rev. **10**, 408] is brought into canonical form. It is shown that a Hamiltonian exists, although it is not uniquely determined by the Lagrangian, for the canonical momentum densities cannot be expressed explicitly in terms of the time derivatives of the field variables. The Hamiltonian contains an arbitrary linear combination of the eight algebraic constraints that exist between the canonical variables at each world point. The constraints and the commutators of the field variables are shown to commute with the Hamiltonian, and the equations of motion can be integrated even in the presence of world-line singularities. The quantized theory is obtained from the canonical formalism in the usual way and its covariance is established. *H. C. Corben* (Pittsburgh, Pa.).

Taub, A. H. A special method for solving the Dirac equations. Rev. Modern Physics **21**, 388–392 (1949).

The Dirac equations in binary spinor formalism are

$$(1)\qquad \begin{cases} g^{\sigma}\left(\frac{\hbar}{i}\frac{\partial}{\partial x^{\sigma}}-\frac{e}{c}\Phi_{\sigma}\right)\psi=-imc\bar{\varphi}, \\ g^{\sigma}\left(\frac{\hbar}{i}\frac{\partial}{\partial x^{\sigma}}+\frac{e}{c}\Phi_{\sigma}\right)\varphi=-imc\bar{\psi}, \end{cases}$$

where Φ_σ is the 4-vector of the external field. These are numerically invariant under a proper Lorentz transformation $x^\sigma \to x^{\sigma *}=L_\tau{}^\sigma x^\tau$, where the $L_\tau{}^\sigma$ are constants. The author's method is to take the $L_\tau{}^\sigma$ to be functions of the x's, and to seek conditions for $L_\tau{}^\sigma$ in order that the transforms φ^*, ψ^* of φ, ψ should be solutions of (1) when φ, ψ are plane-wave solutions of (1) with $\Phi_\sigma=0$. The work is based on that of a previous paper [Physical Rev. (2) **73**, 786–798 (1948); these Rev. **10**, 223]. An example shows that the method, which applies classically, may fail for the Dirac equations when the external field is constant, but, in the case when Φ_σ describes a plane wave, it applies exactly and yields solutions previously discussed by Volkow [Z. Physik **94**, 250–260 (1935); note misprint in footnote 2 of the paper, where 250 appears as 25] and by Sengupta [Bull. Calcutta Math. Soc. **39**, 147–153 (1947); these Rev. **10**, 91]. *H. S. Ruse* (Leeds).

Papapetrou, A. Drehimpuls- und Schwerpunktsatz in der Diracschen Theorie des Elektrons. Prakt. Akad. Athēnōn **15**, 404–417 (1940). (German. Greek summary)

The author constructs a constant antisymmetric second order tensor from the solutions of the Dirac equation for a free electron. This tensor contains as its spatial components the angular momentum of the electron and as its space-time components the space-coordinates of the center of gravity, the latter being defined in terms of the total energy of the electron. *A. H. Taub* (Urbana, Ill.).

Ashauer, S. On the classical equations of motion of radiating electrons. Proc. Cambridge Philos. Soc. **45**, 463–475 (1949).

The author computes the field-energy distribution of a self-accelerating electron obeying the third-order Lorentz-Dirac equations. The energy density is calculated numerically for about 200 field points and a summary of the results of these calculations shows in which region the energy is concentrated. Exact solutions for the motion of an electron disturbed by a pulse of electro-magnetic radiation of infinitely short duration are also given and the total energy and momentum acquired by the electron are computed for the "physical" solution. The results for the "non-physical" solution are also given. *A. H. Taub.*

Bureau, Florent. Sur les équations de la mécanique ondulatoire de l'électron. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 104–142 (1949).

The author derives the Dirac equations for the electron from seven hypotheses. These include postulating equations of the form (1) $\sum_{\beta,\nu} a^k_{\alpha\beta}\partial u_\beta/\partial x^k+\sum_\beta A_{\alpha\beta}u^\beta=0$, where $\alpha, \beta=1, 2, \cdots, n$, and $k=1, 2, 3, 4$; and requiring that (1) with $A_{\alpha\beta}=0$ be totally hyperbolic and its discontinuities have only one velocity of propagation. Finally the assumption that (1) be invariant under the Lorentz group is made. This then differs from the usual derivation in that no reference is made to the Schrödinger equation, but instead a comparison is made with the Maxwell equations. *A. H. Taub* (Urbana, Ill.).

Durand, Émile. Recherches sur la théorie de l'électron de Dirac. J. Math. Pures Appl. (9) **28**, 77–135 (1949).

This paper gives a comprehensive treatment of some of the less well-known aspects of Dirac's theory of the electron. Of interest is the representation of the spinor transformation matrix A [see also W. Pauli, Handbuch der Physik, 2d ed., v. 24, part 1, pp. 83–272, Springer, Berlin, 1933, in particular, p. 221]. Also the concept of wave function potentials, analogous to electrodynamic potentials, is introduced. The wave function potentials satisfy the equation

$$(\partial_0{}^2-k_0{}^2)\Phi_K=0, \qquad K=1, 2, 3, 4,$$

and the Dirac wave functions are related to these by $\psi_K=(\gamma_\rho\partial^\rho-\gamma_0 k_0)i\gamma_4\Phi_K$. A general expression for Φ_K is considered and the mathematical properties of the solutions arising from it are investigated. *C. Kikuchi.*

Conway, Arthur W. Quaternions and quantum mechanics. Pont. Acad. Sci. Acta **12**, 259–277 (1948).

It is well known that there is a correspondence between the theory of quaternions (and bi-quaternions) and the theory of two-component spinors. Thus the Lorentz group may be represented as a group of permutations of quaternions and various Lorentz invariant theories formulated in terms of quaternions. This paper considers the Dirac equation for an electron, formulates it and solves special cases of it in terms of quaternions. *A. H. Taub.*

Brown, G. E. Note on a relation in Dirac's theory of the electron. Proc. Nat. Acad. Sci. U. S. A. **36**, 15–17 (1950).

The author derives a simple relation in Dirac's theory of the electron: $Z\alpha\langle r^{-2}\rangle=k\int fg dr$, where $\langle\ \rangle$ denotes expectation values, Z the nuclear charge, α the fine structure constant, k the quantum number of the total angular momentum, and f, g the Dirac radial wave functions. It allows one to show that the hyperfine structure energy of an electron in a Coulomb field is proportional to $\langle r^{-3}\rangle$. *E. Gora.*

Galanin, A. D. The radiation correction to Dirac's equation. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **19**, 521–534 (1949). (Russian)

The author uses non-covariant perturbation theory for a calculation of the second-order radiative corrections to the equation of motion of an electron in external electric and magnetic fields. The starting-point is standard quantum electrodynamics, with electrons treated according to the Dirac hole theory. The method is essentially the same as that used by French and Weisskopf [Physical Rev. (2) **75**, 1240–1248 (1949)]. The infinite self-energy of the electron appears in the course of the calculation, and the separation of this unobservable self-energy from finite observable effects is not unambiguous in a non-covariant treatment. The discussion and justification of the procedure followed in making the separation is not so complete as that given by French and Weisskopf, although the final results are identical. A treatment of the same problem by Luttinger [Physical Rev. (2) **74**, 893–898 (1948)] is criticised, in the reviewer's opinion wrongly. *F. J. Dyson* (Birmingham).

Starikin, Yu. A. Conservation laws in Dirac's theory. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **19**, 749–752 (1949). (Russian)

The "5-dimensional optics" of Rumer [same vol., 86–94, 207–214 (1949); these Rev. **10**, 580] is a formal method of introducing a 5-dimensional symmetry into the equations of dynamics. Here it is shown that the method leads to conservation laws for momentum, energy and charge, which reduce correctly to the conservation laws of the Dirac electron theory when conditions are appropriately specialised. *F. J. Dyson* (Birmingham).

Johnson, M. H., and Lippmann, B. A. Motion in a constant magnetic field. Physical Rev. (2) **76**, 828–832 (1949).

The motion of a charged particle in a constant magnetic field is treated in both relativistic and nonrelativistic quantum theory. The wave functions are determined by factorization after various constants of the motion are deduced. *A. H. Taub* (Urbana, Ill.).

Watanabe, Satosi. On Dirac's general transformation function. IIb. Progress Theoret. Physics **4**, 1–10 (1949).

See the author's earlier papers [same journal **2**, 71–88 (1947); **3**, 378–390 (1948); these Rev. **10**, 227; **11**, 147]. In this paper he discusses mainly the conditions under which a classical transformation function will exist and be unique. It is found that this is the case only for a restricted class of problems, whereas a quantum-theoretical transformation function can be defined much more generally. Thus there can be little hope of using the classical function fruitfully as a basis for a reformulation of quantum theory. *F. J. Dyson* (Birmingham).

Sawada, Katurô, Takagi, Shûji, and Kobayasi, Minoru. Notes on the self-energy problems. Progress Theoret. Physics **4**, 71–81 (1949).

In quantum electrodynamics there exist two kinds of divergences. First, divergences produced by singular D-functions in the commutation relations of field-quantities; these divergences, like the classical electron self-energy, can be removed by the Dirac λ-limiting process. Second, divergences produced by singular D^1 functions in the vacuum expectation values of products of field-quantities; the D^1 functions express the effect of quantum fluctuations of the fields, and to remove the second class of divergences the Dirac formalism with negative-energy photons is required. The authors here propose a method of removing the second kind of divergences, by replacing the D^1 function by an arbitrary nonsingular function. The method is a generalisation of Dirac's, and also involves the use of negative energy photons. It is not made clear whether the method can be developed into a realistic physical formalism, or whether it is to be regarded merely as a mathematical cut-off device. Also, the method is only discussed in connection with the simple second-order divergences. *F. J. Dyson* (Birmingham).

Fukuda, Hiroshi, Miyamoto, Yonezi, and Tomonaga, Sinitirô. A self-consistent subtraction method in the quantum field theory. II_1. Progress Theoret. Physics **4**, 47–59 (1949).

Fukuda, Hiroshi, Miyamoto, Yoneji, and Tomonaga, Sinitirô. A self-consistent subtraction method in the quantum field theory. II_2. Progress Theoret. Physics **4**, 121–129 (1949).

The second-order radiative corrections to the equation of motion of an electron moving in an external electromagnetic field are calculated; sufficient details are given to enable the reader to reconstruct the whole of the calculation. The starting-point is the covariant quantum electrodynamics of Tomonaga, used in conjunction with the contact transformation method described in a previous paper [T. Tati and S. Tomonaga, same journal **3**, 391–406 (1948); these Rev. **11**, 145]. Infinite mass and charge renormalisation effects are identified and removed in the usual way.

From the point of view of relativistic covariance, the calculation stands half-way between the covariant methods of Schwinger [Physical Rev. (2) **74**, 1439–1461 (1948); these Rev. **10**, 345] and the non-covariant method of French and Weisskopf [Physical Rev. (2) **75**, 1240–1248 (1949)]. The external potential as modified by radiative effects is expressed as an invariant integral independent of the particular space-like surface on which the wave-function of the system is defined. But the integral is then evaluated in a non-covariant way, by specializing the space-like surface to a flat surface and then passing to a 3-dimensional momentum representation. Because of the lack of covariance, the results obtained are formally neither relativistic nor gauge-invariant. For an external electrostatic potential the radiative correction agrees with that found by completely covariant methods; but an external vector potential gives rise to a spurious finite and non-gauge-invariant term, which was also found in previous non-covariant calculations. The authors discuss these and other difficulties of the theory in a simple and clear way. *F. J. Dyson* (Birmingham).

Nambu, Y. Second configuration space and third quantization. Progress Theoret. Physics **4**, 96–98 (1949).

A brief sketch of a new method of manipulating field operators in quantum electrodynamics. Symbols are introduced which represent the operations of adding or removing one factor in a product of operators. These operation-symbols are supposed to bear to the operators the same relation as the operators, which are in fact emission and absorption operators, bear to many-particle wave-functions in configuration space. Hence the name "third quantization." It remains to be seen whether the new notations will prove themselves fruitful; further details will be forthcoming later. *F. J. Dyson* (Birmingham).

Tani, Smio. Note on the formal solution of the Tomonaga-Schwinger equation. Progress Theoret. Physics **4**, 209–217 (1949).

As is well known, the Tomonaga-Schwinger equation $ihc\partial\Psi(\sigma)/\partial\sigma(x)=H(x)\Psi(\sigma)$ can be formally solved by the expression $\Psi(\sigma)=\prod_{\sigma'}^{\sigma}(1-(i/hc)H(x))\Psi(\sigma')$, where the $\prod$ represents a suitably defined continuous product between the surfaces σ' and σ. In this paper the expansion of $\Psi(\sigma)$ as a power-series in H is studied, and various formal identities derived which will be used in subsequent investigations.

F. J. Dyson (Birmingham).

Pauli, W., and Villars, F. On the invariant regularization in relativistic quantum theory. Rev. Modern Physics **21**, 434–444 (1949).

Wentzel [Physical Rev. (2) **74**, 1070–1075 (1948)] showed that despite the success of the Tomonaga-Schwinger-Feynman rules for dealing with the infinities in quantum electrodynamics, there remains some ambiguity in their application. The present authors trace this ambiguity to the indeterminate expressions which arise from products of δ-type singularities and poles on the light cone, and to the possible meaninglessness of the "covariance" of divergent expressions. The present paper avoids these ambiguities by a method which unifies and illuminates the work of previous authors. A relativistically invariant regularization process is introduced by which a field function $F(\kappa)$, involving Δ and D functions, where κ represents the square of the rest mass, is replaced by the "regularized" function $F_R=\int_0^\infty F(\kappa)d\rho(\kappa)$ obtained by taking the Stieltjes integral over a mass distribution $\rho(\kappa)$ satisfying the conditions (1) $\int_0^\infty d\rho(\kappa)=0$, (2) $\int_0^\infty \kappa d\rho(\kappa)=0$. These conditions are sufficient to eliminate the singularities occurring in single particle problems. For single particles, by setting

$$\rho(\kappa)=\delta(\kappa-m^2)+\rho_1(\kappa),$$

where m is the rest mass of the field involved, one passes from the regularized to the actual problem by allowing $\rho_1(\kappa)$ to approach zero for all finite κ. In order that the result of this limiting process be uniquely defined the authors insist (a) that ρ_1 approach zero in such a way that, for any $a>0$, $\int_a^\infty d(|\rho_1(\kappa)|/\kappa)$ also approaches zero; (b) that the same regulator ρ be applied to all parts of F [this condition is essentially the same as Dyson's remark "the rule would not be unambiguous if we were allowed to split the integral into several parts and evaluate . . . them separately," Physical Rev. (2) **75**, 1736–1755 (1949), § VIII; these Rev. **11**, 145]; (c) that any splitting of F into covariant parts be effected only after F has been regularized.

The method is used to justify Schwinger's results for vacuum polarization, photon self-energy, electron self-energy, and the magnetic moment of the electron. In particular, condition (2) above rules out the possibility of finite photon self-energy which Wentzel had shown to be consistent with Schwinger's original method.

A. J. Coleman (Toronto, Ont.).

Rivier, D. Une méthode d'élimination des infinités en théorie des champs quantifiés. Application au moment magnétique du neutron. Helvetica Phys. Acta **22**, 265–318 (1949).

This paper consists of two parts. (i) A general exposition of the methods of Stueckelberg [not yet published in complete form; see G. Wentzel, Rev. Modern Physics **19**, 1–18 (1947); these Rev. **8**, 555] for describing quantum field theories in terms of a Heisenberg S-matrix. The form of the S-matrix is restricted by the three requirements of relativistic invariance, unitarity and causality. However, the precise determination of the S-matrix is not made from these principles alone, but is deduced from equations of motion of conventional Hamiltonian type. The S-matrix calculated from a Hamiltonian automatically satisfies the requirement of causality. The author raises the question, whether causality can be assured by a less restrictive condition. In particular he asserts that a necessary condition for causality is that the elements of the S-matrix should be entirely constructed out of products of functions of a special type which he denotes by D^e. These functions appear automatically starting from the Hamiltonian formulation; they describe the propagation of waves incoming in the past and outgoing in the future. To the reviewer it is not clear to what extent the appearance of D^e functions in the S-matrix is by itself either necessary or sufficient for causality.

(ii) A calculation of the magnetic moment of the neutron, supposed to arise from interaction with a charged pseudoscalar meson field, using perturbation theory to second order in the meson-nucleon coupling. The coupling is assumed to be of the pseudovector form $g\bar{\psi}_N\gamma_5\gamma_a\psi_P(\partial\phi/\partial x_a)$ +complex conjugate. It is found that the moment is infinite, in disagreement with the result of Slotnick and Heitler [Physical Rev. (2) **75**, 1645–1663 (1949)], Luttinger [Helvetica Phys. Acta **21**, 483–496 (1948)], and Case [Physical Rev. (2) **76**, 1–13 (1949)]; the cause of the discrepancy is not clear. Finally, a method due to Stueckelberg [Rivier and Stueckelberg, Physical Rev. (2) **74**, 218, 986 (1948)] is described by which the S-matrix may be modified in such a way as to conserve relativistic invariance, unitarity and causality, while eliminating divergences. The method, generally known as "regularization," has been further investigated by Pauli and others [see the preceding review]; it involves a radical departure from the Hamiltonian description of field-theory. Application of the method makes the neutron moment finite, but the value obtained for it is arbitrary, depending on adjustable parameters in the regularization process.

F. J. Dyson (Birmingham).

Houriet, A., et Kind, A. Classification invariante des termes de la matrice S. Helvetica Phys. Acta **22**, 319–330 (1949).

The authors consider the calculation of the S-matrix for a physical system composed of coupled fields, treating the coupling as a small perturbation. As an example, they apply their method to a charged scalar field u and a neutral scalar field ϕ with the scalar interaction $\epsilon u^*u\phi$. Using conventional perturbation theory, a typical contribution to the S-matrix appears in a form which is not Lorentz invariant; such a contribution, corresponding to a particular sequence of absorptions and emissions of particles in intermediate states, involves explicitly the time-order in which the intermediate events occur. The authors show that the terms can always be grouped together in a simple way, so that the sum of each group depends on the intermediate events in a way which no longer involves their time-order. The sum of each group of terms is Lorentz invariant and of a form well adapted to practical calculations. This work was based upon ideas of Stueckelberg [see the preceding review]. The method is similar to one developed independently by Feynman and described in a paper of the reviewer [Physical Rev. (2) **75**, 486–502 (1949); these Rev. **10**, 418].

F. J. Dyson (Birmingham).

Symonds, N. On the motion of a vector meson in a homogeneous magnetic field. Philos. Mag. (7) **40**, 636–644 (1949).

The equations (1) $\beta_i D_i\psi + K\psi = 0$, where $D_i = \partial/\partial x_i - ie\varphi_i$ and β_i are 10×10 matrices satisfying

$$\beta_i\beta_j\beta_k + \beta_k\beta_j\beta_i = \delta_{ij}\beta_k + \delta_{jk}\beta_i,$$

are solved exactly in the case in which $\varphi_i = H\times\delta_i^2$. The method consists in choosing the β_i appropriately and then solving the second order equations implied by (1).

A. H. Taub (Urbana, Ill.).

Majumdar, R. C., and Gupta, S. The meson fields and the equation of motion of a spinning particle. Physical Rev. (2) **75**, 1788–1793 (1949).

By applying M. Riesz's method of analytic continuation [cf. Acta Math. **81**, 1–223 (1949); these Rev. **10**, 713] to a meson field generated by a point dipole, the authors not only obtain the classical field quantities at points outside the world line but also finite results on the world line itself, representing the reaction of the field on the source particle and agreeing with those of Harish-Chandra [Proc. Roy. Soc. London. Ser. A. **185**, 269–287 (1946); these Rev. **7**, 538]. The equations of motion of the dipole then follow immediately, according to Frenkel [Z. Physik **37**, 243–262 (1926)]. The equations thus obtained are free from singularities and do not contain any arbitrary constants except those which appear as mass and spin angular momentum of the dipole.

L. Hulthén (Stockholm).

Schouten, J. A. On meson fields and conformal transformations. Rev. Modern Physics **21**, 421–424 (1949).

The author lists a number of physical theories which are either invariant under conformal transformations in four variables or in which these transformations play a role.

A. H. Taub (Urbana, Ill.).

Schouten, J. A. Meson fields and conformal geometry. Nederl. Akad. Wetensch. Verslagen, Afd. Natuurkunde **58**, 12–16 (1949). (Dutch)

Cf. the preceding review.

Lipmanov, È. M. On invariant commutation relations and the elimination of auxiliary conditions in the quantum theory of the meson field. Doklady Akad. Nauk SSSR (N.S.) **67**, 627–630 (1949). (Russian)

In the case of the complex vector meson field considered by the author the canonical quantization scheme of Heisenberg and Pauli [Z. Physik **56**, 1–61 (1929)] for the Fourier components of the field is unsatisfactory because the commutation relations are not consistent with the auxiliary conditions which the components must satisfy. To avoid this difficulty the author looks for new commutation relations among the Fourier components which (1) are relativistically invariant and (2) for independent components reduce to the canonical commutation relations. Choosing a special frame of reference in which the auxiliary conditions simply bring about the vanishing of certain Fourier components, the author takes the canonical commutation relations for the nonvanishing components and then puts these relations into a Lorentz covariant form. The resulting commutation relations differ from the canonical ones, but it is possible to eliminate the auxiliary conditions and to obtain the canonical commutation relations by a transformation to new variables.

N. Rosen.

de Broglie, Louis. Sur le calcul classique de l'énergie et de la quantité de mouvement d'un électron purement électromagnétique. C. R. Acad. Sci. Paris **228**, 1264–1268 (1949).

de Broglie, Louis. Sur une forme nouvelle de l'interaction entre les charges électriques et le champ électromagnétique. C. R. Acad. Sci. Paris **229**, 157–161 (1949).

de Broglie, Louis. Nouvelles remarques sur l'interaction entre une charge électrique et le champ électromagnétique. C. R. Acad. Sci. Paris **229**, 269–271 (1949).

In the first paper the author points out that the electromagnetic four-potentials introduced by Wentzel and used by Feynman [Physical Rev. (2) **74**, 939–946 (1948); these Rev. **10**, 222] may be interpreted as due to a charge distribution given by $\sigma(r) = ek_0^2 e^{-k_0 r}/(4\pi r)$ and this in turn has a scalar potential $V(r) = e(1-e^{-k_0 r})/r$, which is the difference of a Coulomb potential and a Yukawa one. It is then verified that such a charge distribution leads to a finite self-energy.

In the second paper the author assumes that the electromagnetic field is composed of two fields: (1) a photon field (with photons of nonvanishing mass and charge e) and (2) a meson field with mesons of charge $e_1 = -e$. He then shows that the distribution of total charge and the scalar potential created by this charge are $\sigma(r) = e\{k_0^2 e^{-k_0 r}/(4\pi r) - \gamma^2 e^{-\gamma r}/(4\pi r)\}$ and $V(r) = e(e^{-\gamma r} - e^{-k_0 r})/r$.

A. H. Taub (Urbana, Ill.).

de Broglie, Louis. Sur la théorie du champ soustractif. C. R. Acad. Sci. Paris **229**, 401–404 (1949).

[Cf. the preceding review.] In this paper the author reinterprets various terms in the field equations which involve the vector potentials of the fields linearly as currents subjected to the action of the total field (i.e., Maxwell plus meson field). With this interpretation he is led to an interaction term in the Hamiltonian which leads to finite self-energies directly without the additional hypotheses used by other authors.

A. H. Taub (Urbana, Ill.).

Gupta, K. K. A note on the spin in Bhabha's theory of elementary particles. Proc. Indian Acad. Sci., Sect. A. **29**, 274–276 (1949).

Bhabha [Rev. Modern Physics **17**, 200–216 (1945); same Proc., Sect. A. **21**, 241–264 (1945); these Rev. **7**, 272, 538; Reports on Progress in Physics **10**, 253–271 (1946)] has discussed possible wave-equations for a particle of finite mass, showing that one irreducible equation corresponds to every irreducible representation $R_s(n, m)$ of the 5-dimensional Lorentz group. The author here proves a conjecture of Bhabha that the spin of the particle corresponding to $R_s(n, m)$ is equal to m.

F. J. Dyson (Birmingham).

Gião, Antonio. Sur la quantification du champ métrique et les interactions particules-champs. I. Application au champ électrique. C. R. Acad. Sci. Paris **230**, 278–280 (1950).

Gião, Antonio. Sur la quantification du champ métrique et les interactions particules-champs. II. Application aux champs magnétique et nucléaire. C. R. Acad. Sci. Paris **230**, 434–436 (1950).